# FINITE ELEMENT SOLUTION FOR THIN SHELLS OF REVOLUTION

## by Z. A. Lu, J. Penzien, and E. P. Popov

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## FOREWORD

The research described in this report, "Finite Element
Solution for Thin Shells of Revolution", was conducted under
the supervision and technical responsibility of Joseph Penzien
and Egor P. Popov, Professors of Civil Engineering, Division
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derivations.

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# I.  INTRODUCTION

.Thin shells of revolution are widely used in flight structures and
their analysis is of great importance to the design engineer.  In such
shells for symmetrical loadings and small displacements, the membrane
stresses and the corresponding elastic displacements can be readily
computed.  However, due to the variations in thickness, ring-like
reinforcements at openings and junctures with the adjoining shells and/or
structures, very important bending stresses develop.  The analysis of such
stresses may be very complex.  In fact, solutions are available only for
the few simplest possible shapes of the meridian.  Also very few solutions
exist for the cases of variable thickness and in some of the solutions
which are available, the thickness variation is prescribed for reasons of
mathematical expediency.  On the other hand, functional and manufacturing
requirements often demand arbitrary shape and thickness variation of the
shell of revolution.  To achieve a practical solution for such a general
problem is the primary purpose of this investigation.

This first report of the general investigation confines its
attention to the elastic analysis of arbitrary shells of revolution
assuming small deformations.  A general solution of this problem has been
obtained by employing finite elements into which any shell of revolution
may be subdivided.  The basic finite element is a truncated conical shell.
Presentation of the detailed analysis of element flexibility, joint loads,
matrix solution of the problem, and examples solved with the aid of an
IBM 7090 computer form the basis of this report.  The developed proce-
dures are quite general and can be applied to any symmetrically loaded

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shell of revolution.  This includes possible variations in shell thickness as well as boundary conditions.

## II.  REVIEW OF MATHEMATICAL FORMULATIONS

A search of literature shows that the general problem of axi-symmetrically loaded shells of revolution has not been completely solved for the case of arbitrary shape and thickness, non-elastic material and large deformations.  However, governing differential equations have been formulated and solutions are available for certain special cases with elastic materials.  These will be summarized below.

H. Reissner and Meissner first formulated the governing equations of shells of revolution based on the classical theory of elasticity which can be easily found in Flügge's book on shells[1].

$$\frac{r_2}{r_1}\chi'' + \left[\frac{r_2}{r_1}\cot\phi \; + \; (\frac{r_2}{r_1})' \; + \; \frac{r_2}{r_1}\frac{K'}{K}\right]\chi'$$

$$- \left[\frac{r_1}{r_2}\cot^2\phi \; + \; \nu \; - \; \nu\frac{K'}{K}\cot\phi\right]\chi = \frac{r_2 r_2 Q_\phi}{K}$$

(A)

$$\frac{r_2}{r_1}(r_2 Q_\phi)'' + \left[\frac{r_2}{r_1}\cot\phi \; + \; (\frac{r_2}{r_1})' \; - \; \frac{r_2}{r_1}\frac{D'}{D}\right](r_2 Q_\phi)'$$

$$- \left[\frac{r_1}{r_2}\cot^2\phi \; - \; \nu \; - \; \nu\frac{D'}{D}\cot\phi\right](r_2 Q_\phi) = -D(1-\nu^2)r_1\chi + P_g(\phi)$$

where $\quad K = \dfrac{Eh^3}{12(1-\nu^2)} \quad , \quad D = \dfrac{Eh}{1-\nu^2}$

Timoshenko in his book gives also the same relations using somewhat different notations[2].

The general procedure of solving Equations (A) consists of eliminating

one or the other of the dependent variables and forming an uncoupled fourth order differential equation. Such an equation is then split, if possible, into two second order differential equations which are then solved. Alternatively, an asymptotic or numerical procedures are used to obtain a solution.

For thin spherical shells of constant thickness, the Reissner-Meissner equations (A) can be solved exactly by the use of hypergeometric series[1][2]; and Zagustin and Young solution[3] using asymptotic integration gives good results for all regions of the shell.

For variable thickness, splitting the two fourth order differential equations into second order equations can be achieved under certain conditions[1]. But asymptotic methods are necessary in general. Rygol[4] solved the resulting asymptotic differential equation in the form $Q_\phi'''' + 4\lambda^4 Q_\phi = 0$ by approximating the thickness function $\lambda$ with two constants. The solution is exact for certain variation of thickness. Kovalenko[5] solved the problem of conical shells of linearly variable thickness when the thickness increases toward the apex.

When the deformations increase, the problem becomes more complicated. E. Reissner[6][7] formulated the governing equations for the case of small deformation but arbitrary rotation as follows.

$$\beta'' + \frac{(rD/\alpha)'}{(rD/\alpha)} \beta' - \left(\frac{\alpha}{r}\right)^2 \cos(\phi + \beta)\left[\sin(\phi + \beta) - \sin\phi\right]$$

$$+ \nu \frac{\alpha}{r} \frac{D'}{D}\left[\sin(\phi + \beta) - \sin\phi\right] + \frac{\nu\alpha}{r}\left[\cos(\phi + \beta) - \cos\phi\right]\phi'$$

$$= \frac{\alpha^2}{rD}\left[\psi \sin(\phi + \beta) - rV \cos(\phi + \beta)\right]$$

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$$\psi'' + \frac{(r/C\alpha)'}{(r/C\alpha)}\psi' - \left\{\frac{\alpha^2}{r^2}\cos^2(\phi+\beta) - \frac{\nu\alpha}{r}\left[\sin(\phi+\beta)(\phi'+\beta')\right.\right.$$

$$\left.\left. + \frac{C'}{C}\cos(\phi+\beta)\right]\right\}\psi = \frac{\alpha^2 C}{r}\left[\cos(\phi+\beta) - \cos\phi\right]$$

$$+ \frac{\nu\alpha}{r}\sin(\phi+\beta)(rV)' + \left\{\frac{\alpha^2}{r^2}\sin(\phi+\beta)\cos(\phi+\beta)\right.$$

$$\left. + \frac{\nu\alpha}{r}\left[\cos(\phi+\beta)(\phi'+\beta') - \frac{C'}{C}\sin(\phi+\beta)\right]\right\}(rV) - \frac{\alpha}{r}(r^2 p_H)'$$

$$- \left[\nu\frac{\alpha^2}{r^2}\cos(\phi+\beta) - \frac{\alpha}{r}\frac{C'}{C}\right](r^2 p_H)$$

(B)

where $\quad D = \dfrac{Eh^3}{12(1-\nu^2)} \qquad C = Eh$

By expanding in series

$$\cos(\phi+\beta) = \cos\phi - \beta\sin\phi - \frac{\beta^2}{2!}\cos\phi \ \ \cdots\cdots$$

$$\sin(\phi+\beta) = \sin\phi + \beta\cos\phi - \frac{\beta^2}{2!}\sin\phi \ \ \cdots\cdots$$

and retaining only powers up to, or in certain expressions, lower than $\beta^2$ he obtained the "small finite deflection theory". The governing equations become

$$\beta'' + \frac{(rD/\alpha)'}{(rD/\alpha)}\beta' - \left[\left(\frac{r'}{r}\right)^2 - \nu\frac{(r'D/\alpha)'}{(rD/\alpha)}\right]\beta + \left[\frac{3}{2}\frac{r'z'}{r^2} - \frac{\nu}{2}\frac{(z'D/\alpha)'}{(rD/\alpha)}\right]\beta^2$$

$$= \frac{\alpha^2}{rD}\left\{\psi\sin\phi - (rV)\cos\phi + \left[\psi\cos\phi + (rV)\sin\phi\right]\beta\right\}$$

(B-a)

$$\psi'' + \frac{(r/C\alpha)'}{(r/C\alpha)}\psi' - \left[\left(\frac{r'}{r}\right)^2 + \nu\frac{(r'/C\alpha)'}{(r/C\alpha)}\right]\psi$$

$$+ \left[2\frac{z'r'}{r^2} + \nu\frac{(z'/C\alpha)'}{(r/C\alpha)}\right]\beta\psi + \nu\frac{z'}{r}\beta'\psi = -\frac{\alpha^2 C}{r}\left[\beta\sin\phi +\right.$$

$$+ \frac{\beta^2}{2}\cos\phi\right] + \left[\frac{z'r'}{r^2} + \nu\frac{(z'/C\alpha)'}{(r/C\alpha)}\right](rV) + \nu\frac{z'}{r}(rV)'$$

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$$+ \left[ \frac{r'^2 - z'^2}{r^2} + \nu \frac{(r'/C\alpha)'}{(r/C\alpha)} \right] \beta\,(rV) + \nu \frac{r'}{r} \beta'\,(rV) + \nu \frac{r'}{r} \beta\,(rV)'$$

$$- \frac{\alpha}{r}\,(r^2 p_H)' - \left[ \nu\,(\frac{\alpha r'}{r^2} - \frac{\alpha z'}{r^2}\,\beta) - \frac{\alpha}{r}\,\frac{C'}{C} \right](r^2 p_H)$$

For the general case, a solution of these equations is very diffi-cult. But for special cases (cylinders, spheres) with constant thickness, asymptotic solutions are possible.

If only the linear terms are retained, the above equations further simplify into those corresponding to the "small deformation theory", different form from that given by (A). The equations take the following form:

$$\beta'' + \frac{(rD/\alpha)'}{(rD/\alpha)}\,\beta' - \left[ (\frac{r'}{r})^2 - \nu \frac{(r'D/\alpha)'}{(rD/\alpha)} \right]\beta = \frac{z'}{(rD/\alpha)}\,\psi - \frac{r'}{(rD/\alpha)}\,(rV)$$

$$\psi'' + \frac{(r/C\alpha)'}{(r/C\alpha)}\,\psi' - \left[ (\frac{r'}{r})^2 + \nu \frac{(r'/C\alpha)'}{(r/C\alpha)} \right]\psi = - \frac{z'}{(r/C\alpha)}\,\beta$$

$$+ \left[ \frac{z'r'}{r^2} + \nu \frac{(z'/C\alpha)'}{(r/C\alpha)} \right](rV) + \nu \frac{z'}{r}\,(rV)' \qquad\text{(B-b)}$$

$$- \left[ \frac{(r/C\alpha)'}{(r/C\alpha)} + \nu \frac{r'}{r} \right](r\alpha p_H) - (r\alpha p_H)'$$

The solution possibility of this set has been discussed before.

Naghdi and De Silva[8][9][10] did some work in detail on E. Reissner's equation. Their solution process made use of complex auxiliary functions and Langer's method of asymptotic integration[11].

Further complication arises when the effect of transverse shear deformation is considered. Following E. Reissner's formulation for small deformations Naghdi[12][13] included transverse shear deformation and derived the following equations for general axisymmetrical shells of revolution.

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$$\beta'' + \left[ (1 - k\lambda^2) \frac{(rD/\alpha)'}{(rD/\alpha)} - k\lambda \left(2\lambda' + \frac{k'}{k}\lambda\right) \right] \beta' - \left\{ (1 - k\lambda^2) \right.$$

$$\left[ \left(\frac{r'}{r}\right)^2 - \nu \frac{(r'D/\alpha)'}{(rD/\alpha)} \right] + \nu k\lambda \left(2\lambda' + \frac{k'}{k}\lambda\right) \left(\frac{r'}{r}\right) \right\} \beta$$

$$= \frac{z'}{(rD/\alpha)} \psi - \frac{r'}{(rD/\alpha)} (rV) + \left(\frac{\lambda}{C}\right) \left\{ - \left[ (r'H)' - \nu\,\psi'' \right] - \right.$$

$$\left[ \frac{(\lambda/C)'}{(\lambda/C)} + \frac{(rD/\alpha)'}{(rD/\alpha)} - \nu \left(\frac{r'}{r}\right) \right] (r'H) + \left[ \nu \frac{(\lambda/C)'}{(\lambda/C)} + \right.$$

$$\left. \nu \frac{(rD/\alpha)'}{(rD/\alpha)} + \left(\frac{r'}{r}\right) \right] \psi' \right\} + \left(\frac{\lambda}{C}\right) \left\{ \left[ (z'V)' - \nu (r\alpha\,p_H)' \right] \right.$$

$$+ \left[ \frac{(\lambda/C)'}{(\lambda/C)} + \frac{(rD/\alpha)'}{(rD/\alpha)} - \nu \left(\frac{r'}{r}\right) \right] (z'V) - \left[ \nu \frac{(\lambda/C)'}{(\lambda/C)} + \right. \qquad \text{(c)}$$

$$\left. + \nu \frac{(rD/\alpha)'}{(rD/\alpha)} - \left(\frac{r'}{r}\right) \right] (r\alpha\,p_H) \right\}$$

$$\psi'' + \frac{(r/C\alpha)'}{(r/C\alpha)} \psi' - \left[ \left(\frac{r'}{r}\right)^2 + \nu \frac{(r'/C\alpha)'}{(r/C\alpha)} \right] \psi = - \frac{z'}{(r/C\alpha)} \beta$$

$$- \frac{k\lambda}{(r/C\alpha)} \left\{ \nu\left(\frac{r}{\alpha}\right) \beta'' + \left[ 2 \left(\frac{r'}{\alpha}\right) + \nu \left(\frac{r}{\alpha}\right)' + \nu \left(\frac{r}{\alpha}\right) \frac{(k\lambda)'}{(k\lambda)} \right] \beta' \right.$$

$$\left. + \left[ \left(\frac{r}{\alpha}\right)\left(\frac{r'}{r}\right)' + \left(\frac{r}{\alpha}\right)'\left(\frac{r'}{r}\right) + \nu \left(\frac{r'}{\alpha}\right)\left(\frac{r'}{r}\right) + \left(\frac{r'}{\alpha}\right) \frac{(k\lambda)'}{(k\lambda)} \right] \beta \right\}$$

$$+ \left[ \frac{z'r'}{r^2} + \nu \frac{(z'/C\alpha)'}{(r/C\alpha)} \right] (rV) + \nu \frac{z'}{r} (rV)' - \left[ \frac{(r/C\alpha)'}{(r/C\alpha)} \right.$$

$$\left. + \nu \frac{r'}{r} \right] (r\alpha\,p_H) - (r\alpha\,p_H)' - \frac{z'}{(r/C\alpha)} \gamma^0_{\xi S}$$

(Note: $\psi = rH$)   $\qquad k = \dfrac{h^2}{12(1-\nu^2)}$

If we set $\lambda$ and $\gamma_{\xi S} = 0$ , the equations reduce to (B-b).

The solution procedure for these equations is much the same as that used earlier by Naghdi for the case without transverse shear effect.

For large deformations and non-linear material, solution can be found only for membrane theory with isotropic incompressible material[17].

Since the available mathematical solutions are extremely complex and since it is almost impossible to incorporate into these solutions

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the general practical range of shells with regard to shape, thickness, material, and magnitude of deformations, an approximate, stepwise, pseudo-elastic finite element solution has been formulated. In a general problem, the actual shell of revolution is considered to be an assemblage of a large number of straight cones each of uniform thickness with the exception of the top (or bottom) element which is a shallow spherical cap of its own uniform thickness. Loads can be applied in small increments for investigations of large deformations and/or inelastic behavior of material. For each step of loading the displacements of the nodal points and the forces in the conical and spherical elements can be calculated by standard matrix operations. The displacements of the nodal points define a new geometry of the assemblage and hence new geometry and structural properties of each element. From the forces in the elements the stresses and strains can be computed which determine whether or not a new set of constants defining the material properties should be assigned to each element for the next increment of load. In this manner, the problems of large deformation and change of material properties after yielding will be treated in subsequent reports. How the material properties vary under biaxial stress conditions is the subject of a separate experimental investigation. By making the size of elements and the load increments sufficiently small, the actual case of smoothly varying shape and thickness as well as gradual yielding of material can be approximated. This report is confined, however, only to the analysis of one load application, i.e., the reported solution is complete for axisymmetrical loading of an arbitrary elastic shell of revolution experiencing small displacements.

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# III.    THEORY OF FINITE ELEMENT SOLUTION

## A.    General Procedure.

The basic principles and procedures employed in the applications
of the finite element method in structural analysis for the case of
small deformations and elastic material have become well known in
recent years and are described in several publications[18][19][20].
The basic steps consist of:

(1) Establishing the stiffness of the structure from the
stiffness (or flexibility) of the individual elements,

(2) Calculation of the forces on, or the displacements of, the
joints of the structure, and

(3) Calculation of the forces and deformations of the elements.

In Step (1), some matrices defining the geometrical relations between
the structure and the elements are used.

In the ordinary structural problems the element flexibilities
are symmetrical with respect to the main diagonal and only three
quantities are required to solve the problem completely.  These
quantities are:

(1) Element stiffness matrix which includes the stiffness
of all elements in a diagonal arrangement,

(2) Loads or displacements imposed on the joints of the
structure, and

(3) The displacement transformation matrix.

However, in the problem considered here, element forces per unit length
are involved and hence unsymmetrical flexibility or stiffness matrices is
obtained.  Therefore, another geometrical matrix which will be called

the "equilibrium matrix" is required in addition to the displacement transformation matrix. Each of these quantities will be discussed in detail in subsequent sections.

## B. Element Flexibilities

1. **Conical Segment.** The displacements of the edges of a conical ring due to forces (including moments) applied on the edges can be obtained from mathematical solutions for conical shells. The expressions derived by Flügge[1] will be used as the basis of our formulation.

Flügge's bending solution of conical shell yields the following relations.

$$Q_s = \frac{1}{s}\left[A_1(\text{ber } y - 2y^{-1}\text{ bei'}y) + A_2(\text{bei } y + 2y^{-1}\text{ ber'}y)\right.$$
$$\left. + B_1(\text{ker } y - 2y^{-1}\text{ kei'}y) + B_2(\text{kei } y + 2y^{-1}\text{ ker'}y)\right]$$

$$N_s = -Q_s \cot \alpha$$

$$N_\theta = -\frac{\cot \alpha}{2s}\left[A_1 \left(y \text{ ber'}y - 2 \text{ ber } y + 4y^{-1}\text{ bei'}y\right)\right.$$
$$+ A_2 \left(y \text{ bei'}y - 2 \text{ bei } y - 4y^{-1}\text{ ber'}y\right)$$
$$+ B_1\left(y \text{ ker'}y - 2 \text{ ker } y + 4y^{-1}\text{ kei'}y\right)$$
$$\left. + B_2 \left(y \text{ kei'}y - 2 \text{ kei } y - 4y^{-1}\text{ ker'}y\right)\right]$$

(I-a)

$$M_s = 2y^{-2}\left\{A_1 \left[y \text{ bei'}y - 2(1-v)(\text{bei } y + 2y^{-1}\text{ ber'}y)\right]\right.$$
$$- A_2 \left[y \text{ ber'}y - 2(1-v)(\text{ber } y - 2y^{-1}\text{ bei'}y)\right]$$
$$+ B_1 \left[y \text{ kei'}y - 2(1-v)(\text{kei } y + 2y^{-1}\text{ ker'}y)\right]$$
$$\left. - B_2 \left[y \text{ ker'}y - 2(1-v)(\text{ker } y - 2y^{-1}\text{ kei'}y)\right]\right\}$$

$$M_\theta = 2y^{-2}\left\{A_1 \left[vy \text{ bei'}y + 2(1-v)(\text{bei } y + 2y^{-1}\text{ ber'}y)\right]\right.$$
$$- A_2 \left[vy \text{ ber'}y + 2(1-v)(\text{ber } y - 2y^{-1}\text{ bei'}y)\right]$$
$$+ B_1 \left[vy \text{ kei'}y + 2(1-v)(\text{kei } y + 2y^{-1}\text{ ker'}y)\right]$$
$$\left. - B_2 \left[vy \text{ ker'}y + 2(1-v)(\text{ker } y - 2y^{-1}\text{ kei'}y)\right]\right\}$$

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$$\chi = \frac{2\sqrt{3(1-\nu^2)} \cot \alpha}{Et^2} \left[ A_1 \left( \text{bei } y + 2y^{-1} \text{ ber'}y \right) \right.$$

$$- A_2 \left( \text{ber } y - 2y^{-1} \text{ bei'}y \right) + B_1 \left( \text{kei } y + 2y^{-1} \text{ ker'}y \right)$$

$$\left. - B_2 \left( \text{ker } y - 2y^{-1} \text{ kei'}y \right) \right]$$

Flügge did not give expressions for $v$ and $w$. But these can be derived from the stress-strain and strain-displacement relations

$$\epsilon_\theta = \frac{1}{Et} \left( N_\theta - \nu N_s \right) \quad ; \quad \epsilon_s = \frac{1}{Et} \left( N_s - \nu N_\theta \right)$$

and $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (II)

$$\epsilon_\theta = \frac{v}{s} + \frac{w \tan \alpha}{s} \quad ; \quad \epsilon_s = \frac{dv}{ds}$$



$t$ = THICKNESS

$\nu$ = POISSON RATIO

$E$ = ELASTIC MODULUS

$\bar{r} = \dfrac{s_j}{s_i}$

$S$ = DISTANCE FROM APEX O

$y = 2\sqrt[4]{3(1-\nu^2)}$

$\sqrt{\dfrac{2 \tan \alpha}{t}} \sqrt{s}$

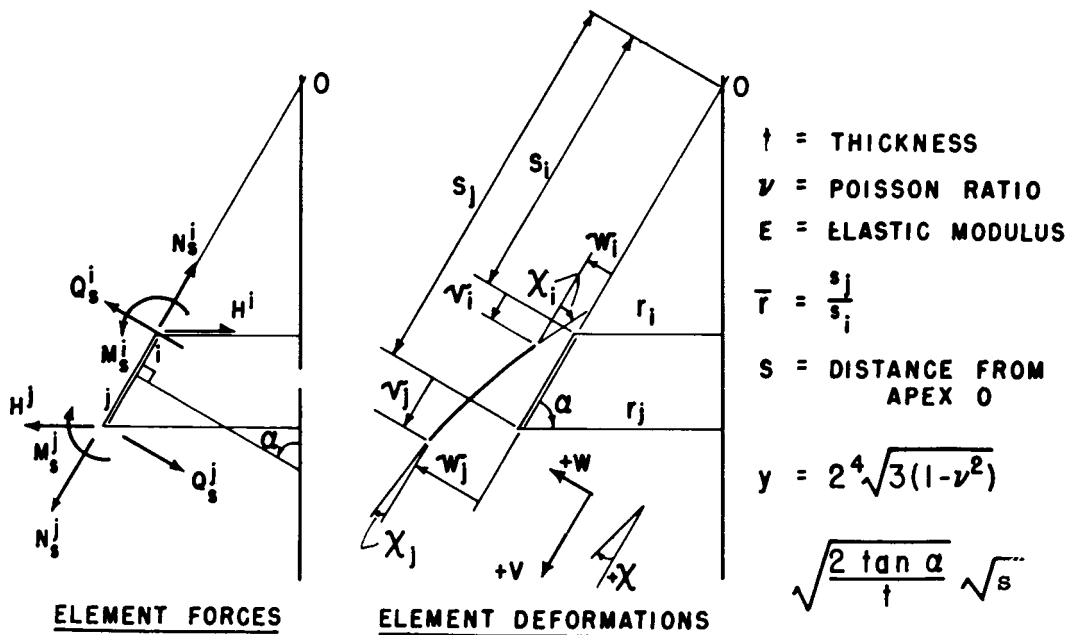ELEMENT FORCES        ELEMENT DEFORMATIONS

FIG. I - CONICAL ELEMENT

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Using the expressions for $N_\theta$ and $N_s$ from (I-a), eliminating and integrating (II), we obtain

$$
\begin{aligned}
v = \frac{2 \cot \alpha}{Et} \Bigg\{ & A_1 \left[ \frac{\nu}{2} \text{ ber } y - (1 + \nu)y^{-1} \text{ bei'}y \right] \\
& + A_2 \left[ \frac{\nu}{2} \text{ bei } y + (1 + \nu)y^{-1} \text{ ber'}y \right] \\
& + B_1 \left[ \frac{\nu}{2} \text{ ker } y - (1 + \nu)y^{-1} \text{ kei'}y \right] \\
& + B_2 \left[ \frac{\nu}{2} \text{ kei } y + (1 + \nu)y^{-1} \text{ ker'}y \right] \Bigg\}
\end{aligned}
$$

(I-b)

$$
\begin{aligned}
w = \frac{\cot^2 \alpha}{Et} \Bigg\{ & A_1 \left[ \text{ber } y - \tfrac{1}{2} y \text{ ber'}y \right] + A_2 \left[ \text{bei } y - \tfrac{1}{2} y \text{ bei'}y \right] \\
& + B_1 \left[ \text{ker } y - \tfrac{1}{2} y \text{ ker'}y \right] + B_2 \left[ \text{kei } y - \tfrac{1}{2} y \text{ kei'}y \right] \Bigg\}
\end{aligned}
$$

In the first of (I-b) integrals of Thompson functions have been removed by using the following identities

$$
\int_0^y y \text{ ber } y \, dy = y \text{ bei'}y \quad , \qquad \int_0^y y \text{ bei } y \, dy = -y \text{ ber'}y
$$

$$
\int_0^y y \text{ ker } y \, dy = y \text{ kei'}y \quad , \qquad \int_0^y y \text{ kei } y \, dy = -y \text{ ker'}y
$$

(1)

The positive directions of the forces and displacements in a conical element are shown in Fig. 1.

It should be noted that Flügge's bending solution of conical shell was obtained from a system in which only horizontal force acts radially at the periphery, the vertical force on the ring is zero. This can be noted from Flügge's comment on the governing differential equation in Ref. (1), and is reflected by the expressions of $Q_s$ and $N_s$ from (I-a), which imply that there is only one force $H$ applied such that

$$
Q_s = -H \sin \alpha \quad , \qquad N_s = H \cos \alpha
$$

(2)

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The expressions $N_s$ and $N_\theta$ given by (I-a) are the resulting membrane forces in the shell due to the application of bending moments and horizontal forces at the edges. If a force $N_s$ is applied at the edge of the ring as a source of disturbance, the induced forces and deformations in the shell cannot be obtained from (I-a) and (I-b) and should be computed from the expressions resulting from a membrane type analysis.

Flügge also gives the solution to the problem of membrane deformation of conical shells[1]. The resulting expressions for the case where no distributed loads are applied to the shell surface are

$$v = \frac{C}{Et} \log s + C_1$$
$$w = -\frac{C}{Et} \cot \alpha (\log s + v) - C_1 \cot \alpha \qquad \text{(I-c)}$$
$$\chi = -\frac{C}{Et} \cot \alpha \cdot \frac{1}{s}$$

where $C$ and $C_1$ are constants of integrations to be determined from a force condition and a displacement condition respectively on the boundary.

For the case of zero distributed loads, $C = N_s \cdot s = N_s^i \cdot s_i = N_s^j \cdot s_j$ .

It must be pointed out that for equilibrium in the vertical direction a force $N_s^j$ applied at edge $j$ has to be balanced by a force $N_s^i$ applied at edge $i$ . The bending moment or horizontal force applied individually at either edge is self-equilibrating. Membrane deformations can only be computed with respect to some reference position which is governed by the constant $C_1$ .

To avoid the coupling of bending and membrane actions and for the reason mentioned in the previous paragraph, we choose $M_s$ , $H$ applied at both edges and $N_s$ applied at the lower edge as the basic element

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force quantities, and arbitrarily assume that the ring is held stationary at the upper edge during stretch, i.e. $v_i = 0$, under the action of membrane force $N_s$ .

Four boundary conditions define the four constants $A_1$ , $A_2$ , $B_1$ , $B_2$ of Eq. (I-a). These are

$$
\begin{array}{llll}
\text{at} \quad y = y_i \; , & \quad M_s = M_i \; , & \quad H = H_i \\
\text{at} \quad y = y_j \; , & \quad M_s = M_j \; , & \quad H = H_j
\end{array}
\tag{3}
$$

Upon substituting the expressions of $A_1$ , $A_2$ , $B_1$ , $B_2$ , which are now functions of $M_i$ , $M_j$ , $H_i$ , $H_j$ into Eqs.(I-a) and (I-b) we obtain the force and deformation quantities as functions of $M_i$ , $M_j$ , $H_i$ , $H_j$ and of Thompson functions and their first derivatives. The algebraic manipulation is lengthy and tedious. Only the resulting expressions in symbolic form will be given below.

Before we present the resulting force and deformation expressions, we need to define a few more quantities.

To correspond with the horizontal force H, there must be a horizontal displacement $\delta$ . From simple geometry

$$
\delta = w \sin \alpha + v \cos \alpha
\tag{4}
$$

We see that the positive direction of $\delta$ is outward.

We then find that the deformations $\chi_i$ , $\chi_j$ , $\delta_i$ , $\delta_j$ as defined by Flügge's solution are not all in the same direction as the corresponding forces $M_i$ , $M_j$ , $H_i$ , $H_j$ . Also since the membrane force $N_j$ and $N_i$ exist simultaneously we must define the corresponding deformation in the same sense. (From now on we abbreviate the symbol $N_s^i$ and $N_s^j$ as $N_i$ and $N_j$ where $N_i = \bar{r} \, N_j$ .) Thus, we introduce the following as the basic element deformation quantities corresponding to the basic

element force quantities we have chosen.

$$\bar{\chi}_i = -\chi_i$$

$$\bar{\chi}_j = \chi_j$$

$$\bar{\delta}_i = -\delta_i \qquad (5)$$

$$\bar{\delta}_j = \delta_j$$

$$e = v_j - v_i$$

The deformations as a result of the application of the forces $M_i$ , $M_j$ , $H_i$ , $H_j$ are

$$
\begin{aligned}
\bar{\chi}_i = -Cp1 \Big\{ & \Big[ a_{12}b_2(y_i) - a_{22}b_1(y_i) + b_{12}k_2(y_i) - b_{22}k_1(y_i) \Big] M_i \\
& + \Big[ a_{14}b_2(y_i) - a_{24}b_1(y_i) + b_{14}k_2(y_i) - b_{24}k_1(y_i) \Big] M_j \\
& + \Big[ a_{11}b_2(y_i) - a_{21}b_1(y_i) + b_{11}k_2(y_i) - b_{21}k_1(y_i) \Big] H_i \\
& + \Big[ a_{13}b_2(y_i) - a_{23}b_1(y_i) + b_{13}k_2(y_i) - b_{23}k_1(y_i) \Big] H_j \Big\}
\end{aligned}
$$

$$
\begin{aligned}
\bar{\chi}_j = Cp1 \Big\{ & \Big[ a_{12}b_2(y_j) - a_{22}b_1(y_j) + b_{12}k_2(y_j) - b_{22}k_1(y_j) \Big] M_i \\
& + \Big[ a_{14}b_2(y_j) - a_{24}b_1(y_j) + b_{14}k_2(y_j) - b_{24}k_1(y_j) \Big] M_j \\
& + \Big[ a_{11}b_2(y_j) - a_{21}b_1(y_j) + b_{11}k_2(y_j) - b_{21}k_1(y_j) \Big] H_i \\
& + \Big[ a_{13}b_2(y_j) - a_{23}b_1(y_j) + b_{13}k_2(y_j) - b_{23}k_1(y_j) \Big] H_j \Big\}
\end{aligned}
$$

$$
\begin{aligned}
\bar{\delta}_i = -Cp3 \Big\{ & \Big[ a_{12}b_5(y_i) + a_{22}b_6(y_i) + b_{12}k_5(y_i) + b_{22}k_6(y_i) \Big] M_i \\
& + \Big[ a_{14}b_5(y_i) + a_{24}b_6(y_i) + b_{14}k_5(y_i) + b_{24}k_6(y_i) \Big] M_j \\
& + \Big[ a_{11}b_5(y_i) + a_{21}b_6(y_i) + b_{11}k_5(y_i) + b_{21}k_6(y_i) \Big] H_i \\
& + \Big[ a_{13}b_5(y_i) + a_{23}b_6(y_i) + b_{13}k_5(y_i) + b_{23}k_6(y_i) \Big] H_j \Big\}
\end{aligned}
$$

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$$\bar{\delta}_j = Cp3 \left\{ \left[ a_{12}b_5(y_j) + a_{22}b_6(y_j) + b_{12}k_5(y_j) + b_{22}k_6(y_j) \right] M_i \right.$$

$$+ \left[ a_{14}b_5(y_j) + a_{24}b_6(y_j) + b_{14}k_5(y_j) + b_{24}k_6(y_j) \right] M_j$$

$$+ \left[ a_{11}b_5(y_j) + a_{21}b_6(y_j) + b_{11}k_5(y_j) + b_{21}k_6(y_j) \right] H_i$$

$$+ \left. \left[ a_{13}b_5(y_j) + a_{23}b_6(y_j) + b_{13}k_5(y_j) + b_{23}k_6(y_j) \right] H_j \right\}$$

(III-a)

$$e = Cp2 \left\{ \left[ a_{12}\left\langle b_9(y_j) - b_9(y_i) \right\rangle + a_{22}\left\langle b_{10}(y_j) - b_{10}(y_i) \right\rangle \right. \right.$$

$$+ b_{12}\left\langle k_9(y_j) - k_9(y_i) \right\rangle + b_{22}\left\langle k_{10}(y_j) - k_{10}(y_i) \right\rangle \right] M_i$$

$$+ \left[ a_{14}\left\langle b_9(y_j) - b_9(y_i) \right\rangle + a_{24}\left\langle b_{10}(y_j) - b_{10}(y_i) \right\rangle \right.$$

$$+ \left. b_{14}\left\langle k_9(y_j) - k_9(y_i) \right\rangle + b_{24}\left\langle k_{10}(y_j) - k_{10}(y_i) \right\rangle \right] M_j$$

$$+ \left[ a_{11}\left\langle b_9(y_j) - b_9(y_i) \right\rangle + a_{21}\left\langle b_{10}(y_j) - b_{10}(y_i) \right\rangle \right.$$

$$+ \left. b_{11}\left\langle k_9(y_i) - k_9(y_i) \right\rangle + b_{21}\left\langle k_{10}(y_j) - k_{10}(y_i) \right\rangle \right] H_i$$

$$+ \left[ a_{13}\left\langle b_9(y_j) - b_9(y_i) \right\rangle + a_{23}\left\langle b_{10}(y_j) - b_{10}(y_i) \right\rangle \right.$$

$$+ \left. \left. b_{13}\left\langle k_9(y_j) - k_9(y_i) \right\rangle + b_{23}\left\langle k_{10}(y_j) - k_{10}(y_i) \right\rangle \right] H_j \right\}$$

where $Cp1 = \dfrac{2\sqrt{3(1-v^2)} \cot \alpha}{Et^2}$ , $Cp2 = \dfrac{\cot \alpha}{Et}$ , $Cp3 = \dfrac{\cot \alpha \cdot \cos \alpha}{Et}$

$$a_{11} = -\sin \alpha \cdot s_i \frac{d_{11}}{\Delta} \; ; \quad a_{12} = -\frac{y_i^2}{2}\frac{d_{12}}{\Delta} \; ; \quad a_{13} = -\sin \alpha \cdot s_j \frac{d_{13}}{\Delta} \; ; \quad a_{14} = -\frac{y_j^2}{2}\frac{d_{14}}{\Delta}$$

$$a_{21} = \sin \alpha \cdot s_i \frac{d_{21}}{\Delta} \; ; \quad a_{22} = \frac{y_i^2}{2}\frac{d_{22}}{\Delta} \; ; \quad a_{23} = \sin \alpha \cdot s_j \frac{d_{23}}{\Delta} \; ; \quad a_{24} = \frac{y_j^2}{2}\frac{d_{24}}{\Delta}$$

$$b_{11} = -\sin \alpha \cdot s_i \frac{d_{31}}{\Delta} \; ; \quad b_{12} = -\frac{y_i^2}{2}\frac{d_{32}}{\Delta} \; ; \quad b_{13} = -\sin \alpha \cdot s_j \frac{d_{33}}{\Delta} \; ; \quad b_{14} = -\frac{y_j^2}{2}\frac{d_{34}}{\Delta}$$

$$b_{21} = \sin \alpha \cdot s_i \frac{d_{41}}{\Delta} \; ; \quad b_{22} = \frac{y_i^2}{2}\frac{d_{42}}{\Delta} \; ; \quad b_{23} = \sin \alpha \cdot s_j \frac{d_{43}}{\Delta} \; ; \quad b_{24} = \frac{y_j^2}{2}\frac{d_{44}}{\Delta}$$

$$\Delta = \begin{vmatrix} b_1(y_i) & b_2(y_i) & k_1(y_i) & k_2(y_i) \\ b_4(y_i) & -b_3(y_i) & k_4(y_i) & -k_3(y_i) \\ b_1(y_j) & b_2(y_j) & k_1(y_j) & k_2(y_j) \\ b_4(y_j) & -b_3(y_j) & k_4(y_j) & -k_3(y_j) \end{vmatrix}$$

$d_{ji}$ = minors of item $ij$ in the determinant $\Delta$.

$$b_1(y) = \text{ber } y - 2y^{-1} \text{ bei'}y$$

$$b_2(y) = \text{bei } y + 2y^{-1} \text{ ber'}y$$

$$k_1(y) = \text{ker } y - 2y^{-1} \text{ kei'}y$$

$$k_2(y) = \text{kei } y + 2y^{-1} \text{ ker'}y$$

$$b_3(y) = y \text{ ber'}y - 2(1-\nu) \, b_1(y)$$

$$b_4(y) = y \text{ bei'}y - 2(1-\nu) \, b_2(y)$$

$$k_3(y) = y \text{ ker'}y - 2(1-\nu) \, k_1(y)$$

$$k_4(y) = y \text{ kei'}y - 2(1-\nu) \, k_2(y)$$

$$b_5(y) = -\tfrac{1}{2}\left[ y \text{ ber'}y - 2(1+\nu) \, b_1(y) \right]$$

$$b_6(y) = -\tfrac{1}{2}\left[ y \text{ bei'}y - 2(1+\nu) \, b_2(y) \right]$$

$$k_5(y) = -\tfrac{1}{2}\left[ y \text{ ker'}y - 2(1+\nu) \, k_1(y) \right]$$

$$k_6(y) = -\tfrac{1}{2}\left[ y \text{ kei'}y - 2(1+\nu) \, k_2(y) \right]$$

$$b_9(y) = \nu \text{ ber } y - 2(1+\nu) \, y^{-1} \text{ bei'}y$$

$$b_{10}(y) = \nu \text{ bei } y + 2(1+\nu) \, y^{-1} \text{ ber'}y$$

$$k_9(y) = \nu \text{ ker } y - 2(1+\nu) \, y^{-1} \text{ kei'}y$$

$$k_{10}(y) = \nu \text{ kei } y + 2(1+\nu) \, y^{-1} \text{ ker'}y$$

From (I-c) with $C = N_j s_j$ and with the condition that $v_i = 0$, we find the deformations due to a pair of membrane force $N_j$ and $\bar{r} N_j$ applied at edges $j$ and $i$.

$$\bar{\chi}_i = \frac{\bar{r} \cot \alpha}{Et} \cdot N_j$$

$$\bar{\chi}_j = \frac{- \cot \alpha}{Et} \cdot N_j$$

$$\bar{\delta}_i = \nu \cdot \frac{s_j \cos \alpha}{Et} N_j \qquad\qquad\text{(III-b)}$$

$$\bar{\delta}_j = -\nu \cdot \frac{s_j \cos \alpha}{Et} N_j$$

$$e = \frac{s_j}{Et} \log (\bar{r}) \cdot N_j$$

With Eq. (III-a) and (III-b) we developed the relationship:

$$\begin{Bmatrix} \bar{\chi}_i \\ \bar{\chi}_j \\ \bar{\delta}_i \\ \bar{\delta}_j \\ e \end{Bmatrix} = \begin{bmatrix} f_{11} & f_{12} & \cdots\cdots & f_{15} \\ f_{21} & f_{22} & \cdots\cdots & f_{25} \\ f_{31} & & & \\ f_{41} & & & \\ f_{51} & \cdots\cdots\cdots & & f_{55} \end{bmatrix} \cdot \begin{Bmatrix} M_i \\ M_j \\ H_i \\ H_j \\ N_j \end{Bmatrix} \qquad (6)$$

or in the matrix notation

$$v = f \cdot s$$

$f$ is the flexibility matrix of the conical element and has a dimension of 5x5. The inverse of $f$, called the stiffness matrix of the element satisfies the relation

$$S = k \, v .$$

Once the edge forces $M_i$, $M_j$, $H_i$, $H_j$, and $N_j$ are known, the forces and deformations at any point in the conical ring can be found from Eqs. (I-a) and (I-b) together with the relation

$$\begin{Bmatrix} A_1 \\ A_2 \\ B_1 \\ B_2 \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \end{bmatrix} \cdot \begin{Bmatrix} H_i \\ M_i \\ H_j \\ M_j \end{Bmatrix} \tag{7}$$

and from (I-c) supplemented by

$$C_1 = -\frac{N_j s_j}{Et} \log s_i$$

$$N_s = \frac{N_j s_j}{s} \tag{8}$$

$$N_\theta = 0$$

Referring back to the shell solutions from which Eq. (6) was developed we see that force quantities are all in units of pounds or pound-inches per unit length. Hence the  f  matrix so obtained is not symmetrical with respect to the main diagonal.  Items of the matrix representing deformations due to forces on the same edge of the ring are symmetrical, while those representing effects of forces across the edges are not.  The latter bear a ratio of $\bar{r}$ to each other which reflects the ratio of the circumferences between the two edges of the ring.  However, it can be easily shown that items of  f  matrix do satisfy Betti's law.  We could use the total forces around an edge as basic force quantities and obtain a symmetrical  f  matrix.  But since at later stages we shall make frequent use of the expressions from shell solutions which involve forces per unit length, we choose to retain its present form.

The accuracy of the  f  matrix, judged from the values of the quantities on opposite sides of the main diagonal, has been investigated

over a wide range of thickness, lengths, angles and radii of cones. It was found that the first 4x4 portion is nearly perfect for almost any case provided the length of the element is not smaller than the thickness, but the fifth column and fifth row which depend on different mathematical functions diverge more and more as the geometry of the cone approaches extreme cases. The first 4x4 submatrix, representing entirely the bending effect, is valid as long as $\tan \alpha$ is finite, but discrepancies between items on opposite sides of the diagonal increase as $\alpha$ approaches $90^\circ$; in such cases the expressions for cylindrical shells should be used. To reduce discrepancies between quantities of the fifth column and the fifth row, it is recommended that for $\alpha < 30^\circ$ (since for small $\alpha$ plate action predominates) quantities of the fifth row be used to establish those of the fifth column (by multiplying with appropriate factors), while for $\alpha > 60^\circ$ quantities of the fifth column be used to establish those of the fifth row.

For conical rings with larger periphery on top, $\alpha$ is greater than $90^\circ$ and $\cos \alpha$ is negative. In this case, we shall take the larger slant distance as $s_i$, the smaller slant distance as $s_j$ $(\bar{r} < 1)$, and we can use the same expression to obtain the flexibility matrix of the element, provided a change of sign is applied to the values thus obtained for all the third and fourth column quantities $(f_{ij}$, $i = 1$ to $5$, $j = 3,4)$ and to the last three of the fifth column quantities $(f_{35}$, $f_{45}$, $f_{55})$. This flexibility matrix after inversion will represent the stiffness of the conical ring.

2. <u>Cylindrical Segment.</u>    For the cylindrical ring, the effects of the bending forces and the membrane force also should be treated separately.  Both the bending solution and membrane solution for axisymmetrical cylinders are available in Ref. (1) and (2).  Using the notations and sign convention by Flügge[1] (Fig. 2), we established, after a considerable amount of manipulation, the required relationships.

With a force N lbs. per inch applied at one end of a cylinder, there must be an equal force N at the other end.  The deformations due to membrane force N are

$$w = -\frac{\nu\,Na}{Et}$$

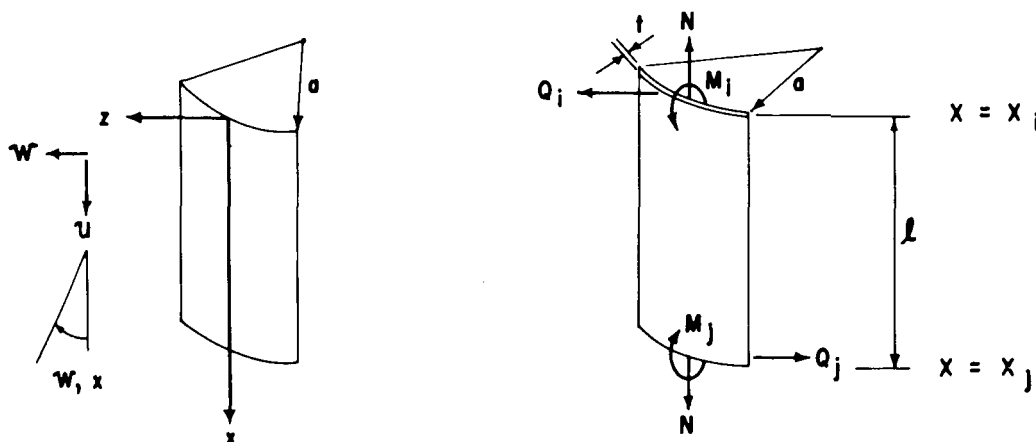$$e = u_j - u_i = \frac{N(x_j - x_i)}{Et}$$

$$\frac{dw}{dx} = 0$$

(IV-a)



FIG. 2 - FORCES AND DEFORMATIONS IN
A CYLINDRICAL ELEMENT

The bending solution of **axi**symmetrical cylindrical sections yields

$$M_x = \frac{2K\,\kappa^2}{a^2}\left[e^{-\kappa\xi}(C_1\sin\kappa\xi - C_2\cos\kappa\xi)\right.$$

$$\left. - e^{\kappa\xi}(C_3\sin\kappa\xi - C_4\cos\kappa\xi)\right]$$

$$Q_x = \frac{2K\,\kappa^3}{a^3}\left\{e^{-\kappa\xi}\left[(C_1+C_2)\cos\kappa\xi - (C_1-C_2)\sin\kappa\xi\right]\right.$$

$$\left. - e^{\kappa\xi}\left[(C_3-C_4)\cos\kappa\xi + (C_3+C_4)\sin\kappa\xi\right]\right\} \qquad (\text{IV-b})$$

$$w = e^{-\kappa\xi}\left[C_1\cos\kappa\xi + C_2\sin\kappa\xi\right] + e^{\kappa\xi}\left[C_3\cos\kappa\xi + C_4\sin\kappa\xi\right]$$

$$\chi = \frac{dw}{dx} = \frac{1}{a}\frac{dw}{d\xi} = -\frac{\kappa}{a}e^{-\kappa\xi}\left[(C_1-C_2)\cos\kappa\xi + (C_1+C_2)\sin\kappa\xi\right]$$

$$+\frac{\kappa}{a}e^{\kappa\xi}\left[(C_3+C_4)\cos\kappa\xi - (C_3-C_4)\sin\kappa\xi\right]$$

$$u = -\frac{\nu}{2\kappa}\left\{e^{-\kappa\xi}\left[(-C_1-C_2)\cos\kappa\xi + (C_1-C_2)\sin\kappa\xi\right]\right.$$

$$\left. + e^{\kappa\xi}\left[(C_3-C_4)\cos\kappa\xi + (C_3+C_4)\sin\kappa\xi\right]\right\} + C$$

where $K = \dfrac{Et^3}{12(1-\nu^2)}$ , $\kappa^4 = 3(1-\nu^2)\dfrac{a^2}{t^2}$ , $\xi = \dfrac{x}{a}$

$C_1$ , $C_2$ , $C_3$ , $C_4$ , and $C$ are constants of integration. The constants $C_1$ , $C_2$ , $C_3$ , $C_4$ can be determined by defining two force quantities at each end of the ring, and the constant $C$ will drop out when only the expression of $e = u_j - u_i$ is wanted.

To conform to the particular force and deformation system adopted as standard for our process (see Fig. 1 and Eqs. (5) ), we apply the four conditions

$$\text{at } \xi = \xi_i = 0 \qquad M_x = M_i$$

$$\text{at } \xi = \xi_i = 0 \qquad Q_x = -H_i$$

$$\text{at } \xi = \xi_j = \frac{\ell}{a} \qquad M_x = M_j \tag{9}$$

$$\text{at } \xi = \xi_j = \frac{\ell}{a} \qquad Q_x = -H_j$$

To obtain expressions of $C_1$, $C_2$, $C_3$, $C_4$ in terms of $M_i$, $M_j$, $H_i$, $H_j$.

Upon substitution of these expressions into (IV-b), there result (reference to Eq. (5) ):

$$\bar{\chi}_i = \frac{K}{a} \left\{ (C_{11} - C_{21} - C_{31} - C_{41}) \cdot M_i + (C_{12} - C_{22} - C_{32} - C_{42}) \cdot M_j \right.$$
$$\left. + (C_{13} - C_{23} - C_{33} - C_{43}) \cdot H_i + (C_{14} - C_{24} - C_{34} - C_{44}) \cdot H_j \right\}$$

$$\bar{\chi}_j = \frac{K}{a} \left\{ \left[ -e^{-\omega}(\cos \omega + \sin \omega) C_{11} + e^{-\omega}(\cos \omega - \sin \omega) C_{21} \right. \right.$$
$$\left. + e^{\omega}(\cos\omega - \sin \omega) C_{31} + e^{\omega}(\cos \omega + \sin \omega) C_{41} \right] \cdot M_i$$
$$+ \left[ -e^{-\omega}(\cos \omega + \sin \omega) C_{12} + e^{-\omega}(\cos \omega - \sin \omega) C_{22} \right.$$
$$\left. + e^{\omega}(\cos \omega - \sin \omega) C_{32} + e^{\omega}(\cos \omega + \sin \omega) C_{42} \right] \cdot M_j$$
$$+ \left[ -e^{-\omega}(\cos \omega + \sin \omega) C_{13} + e^{-\omega}(\cos \omega - \sin \omega) C_{23} \right.$$
$$\left. + e^{\omega}(\cos \omega - \sin \omega) C_{33} + e^{\omega}(\cos \omega + \sin \omega) C_{43} \right] \cdot H_i$$
$$+ \left[ -e^{-\omega}(\cos \omega + \sin \omega) C_{14} + e^{-\omega}(\cos \omega - \sin \omega) C_{24} \right.$$
$$\left. + e^{\omega}(\cos \omega - \sin \omega) C_{34} + e^{\omega}(\cos \omega + \sin \omega) C_{44} \right] \cdot H_j \right\}$$

$$\bar{\delta}_i = (-C_{11} - C_{31}) \cdot M_i + (-C_{12} - C_{32}) \cdot M_j + (-C_{13} - C_{33}) \cdot H_i + (-C_{14} - C_{34}) \cdot H_j$$

$$\bar{\delta}_j = (e^{-\omega}\cos \omega \cdot C_{11} + e^{-\omega}\sin \omega \cdot C_{21} + e^{\omega}\cos \omega \cdot C_{31} + e^{\omega}\sin \omega \cdot C_{41}) \cdot M_i$$
$$+ (e^{-\omega}\cos \omega \cdot C_{12} + e^{-\omega}\sin \omega \cdot C_{22} + e^{\omega}\cos \omega \cdot C_{32} + e^{\omega}\sin \omega \cdot C_{42}) \cdot M_j$$
$$+ (e^{-\omega}\cos \omega \cdot C_{13} + e^{-\omega}\sin \omega \cdot C_{23} + e^{\omega}\cos \omega \cdot C_{33} + e^{\omega}\sin \omega \cdot C_{43}) \cdot H_i$$
$$+ (e^{-\omega}\cos \omega \cdot C_{14} + e^{-\omega}\sin \omega \cdot C_{24} + e^{\omega}\cos \omega \cdot C_{34} + e^{\omega}\sin \omega \cdot C_{44}) \cdot H_j$$

$$e = \frac{\nu}{2\kappa}\left\{\left[e^{-\omega}(\cos\omega - \sin\omega)-1\right]\cdot C_{13} + \left[e^{-\omega}(\cos\omega + \sin\omega)-1\right]\cdot C_{23}\right.$$

$$\left. - \left[e^{\omega}(\cos\omega + \sin\omega)-1\right]\cdot C_{33} + \left[e^{\omega}(\cos\omega - \sin\omega)-1\right]\cdot C_{43}\right\}\cdot H_i$$

$$+ \frac{\nu}{2\kappa}\left\{\left[e^{-\omega}(\cos\omega - \sin\omega)-1\right]\cdot C_{14} + \left[e^{-\omega}(\cos\omega + \sin\omega)-1\right]\cdot C_{24}\right.$$

$$\left. - \left[e^{\omega}(\cos\omega + \sin\omega)-1\right]\cdot C_{34} + \left[e^{\omega}(\cos\omega - \sin\omega)-1\right]\cdot C_{44}\right\}\cdot H_j$$

where $\omega = \kappa\frac{\ell}{a}$, $C_{11}$, $C_{12}$ ....... $C_{43}$, $C_{44}$ satisfy the relations

$$C_1 = C_{11}\cdot M_i + C_{12}\cdot M_j + C_{13}\cdot H_i + C_{14}\cdot H_j$$

$$C_2 = C_{21}\cdot M_i + C_{22}\cdot M_j + C_{23}\cdot H_i + C_{24}\cdot H_j$$

$$C_3 = C_{31}\cdot M_i + C_{32}\cdot M_j + C_{33}\cdot H_i + C_{34}\cdot H_j \tag{10}$$

$$C_4 = C_{41}\cdot M_i + C_{42}\cdot M_j + C_{43}\cdot H_i + C_{44}\cdot H_j$$

$$C_{11} = \frac{D_{11}}{\Delta}\frac{a^2}{2K\kappa^2}, \quad C_{12} = -\frac{D_{21}}{\Delta}\frac{a^2}{2K\kappa^2}, \quad C_{13} = \frac{D_{31}}{\Delta}\frac{a^3}{2K\kappa^3}, \quad C_{14} = -\frac{D_{41}}{\Delta}\frac{a^3}{2K\kappa^3}$$

$$C_{21} = -\frac{D_{12}}{\Delta}\frac{a^2}{2K\kappa^2}, \quad C_{22} = \frac{D_{22}}{\Delta}\frac{a^2}{2K\kappa^2}, \quad C_{23} = -\frac{D_{32}}{\Delta}\frac{a^3}{2K\kappa^3}, \quad C_{24} = \frac{D_{42}}{\Delta}\frac{a^3}{2K\kappa^3}$$

$$C_{31} = \frac{D_{13}}{\Delta}\frac{a^2}{2K\kappa^2}, \quad C_{32} = -\frac{D_{23}}{\Delta}\frac{a^2}{2K\kappa^2}, \quad C_{33} = \frac{D_{33}}{\Delta}\frac{a^3}{2K\kappa^3}, \quad C_{34} = -\frac{D_{43}}{\Delta}\frac{a^3}{2K\kappa^3}$$

$$C_{41} = -\frac{D_{14}}{\Delta}\frac{a^2}{2K\kappa^2}, \quad C_{42} = \frac{D_{24}}{\Delta}\frac{a^2}{2K\kappa^2}, \quad C_{43} = -\frac{D_{34}}{\Delta}\frac{a^3}{2K\kappa^3}, \quad C_{44} = \frac{D_{44}}{\Delta}\frac{a^3}{2K\kappa^3}$$

$$\Delta = \left[2\sin\omega + (e^{\omega} - e^{-\omega})\right]\cdot\left[2\sin\omega - (e^{\omega} - e^{-\omega})\right]$$

$$D_{11} = \cos 2\omega - \sin 2\omega - e^{2\omega}$$

$$D_{21} = -(\cos\omega + \sin\omega)(e^{\omega} - e^{-\omega}) - 2\sin\omega\cdot e^{\omega}$$

$$D_{31} = e^{2\omega} - 1 - \sin 2\omega$$

$$D_{41} = \cos\omega(e^{\omega} - e^{-\omega}) - 2\sin\omega\cdot e^{\omega}$$

$$D_{12} = (\cos \omega - \sin \omega)^2 + 2 \sin^2 \omega - e^{2\omega}$$

$$D_{22} = -\cos \omega (e^{\omega} - e^{-\omega}) - \sin \omega (e^{\omega} + e^{-\omega})$$

$$D_{32} = 2 \sin^2 \omega$$

$$D_{42} = -\sin \omega (e^{\omega} - e^{-\omega})$$

$$D_{13} = -e^{-2\omega} + \sin 2\omega + \cos 2\omega$$

$$D_{23} = (\cos \omega - \sin \omega)(e^{\omega} - e^{-\omega}) + 2 \sin \omega \cdot e^{-\omega}$$

$$D_{33} = 1 - e^{-2\omega} - \sin 2\omega$$

$$D_{43} = \cos \omega (e^{\omega} - e^{-\omega}) - 2 \sin \omega \cdot e^{-\omega}$$

$$D_{14} = e^{-2\omega} - 2 \sin^2 \omega - \sin 2\omega - 1$$

$$D_{24} = -\cos \omega (e^{\omega} - e^{-\omega}) - \sin \omega (e^{\omega} + e^{-\omega})$$

$$D_{34} = 2 \sin^2 \omega$$

$$D_{44} = -\sin \omega (e^{\omega} - e^{-\omega})$$

Equations (IV-a) yield

$$\bar{\chi}_i = 0$$

$$\bar{\chi}_j = 0$$

$$\bar{\delta}_i = -w_i = \frac{\nu a}{Et} N \qquad (11)$$

$$\bar{\delta}_j = w_j = -\frac{\nu a}{Et} N$$

$$e = \frac{\ell}{Et} N$$

From Eqs. (10) and (11), the flexibility matrix of a cylindrical ring can be obtained such that

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$$
\begin{Bmatrix} \bar{\chi}_i \\ \bar{\chi}_j \\ \bar{\delta}_i \\ \bar{\delta}_j \\ e \end{Bmatrix} = \begin{bmatrix} f \end{bmatrix} \cdot \begin{Bmatrix} M_i \\ M_j \\ H_i \\ H_j \\ N \end{Bmatrix} \qquad (12)
$$

in which $f_{15}$ , $f_{25}$ , $f_{51}$ , $f_{52}$ are known to be zeros. The stiffness of the cylindrical ring is obtained by inversion of the $f$ matrix.

3. <u>Spherical Cap.</u>    Flügge and Timoshenko both present the asymptotic solution for bending of a shallow spherical shell in their books[1][2].  The resulting general expressions are given below with the notations shown in Fig. 3.

$$w = C_1 \text{ ber } x + C_2 \text{ bei } x + C_3 \text{ kei } x - C_4 \text{ ker } x + C_6$$

$$v = -\frac{\ell}{a}(1+\nu)\left\{ C_1 \text{ bei'}x - C_2 \text{ ber'}x - C_3 \text{ ker'}x \right.$$
$$\left. - C_4 \text{ kei'}x - \frac{C_5}{x} + C_6 \frac{x}{(1+\nu)} \right\}$$

$$\chi = \frac{1}{\ell}(C_1 \text{ ber'}x + C_2 \text{ bei'}x + C_3 \text{ kei'}x - C_4 \text{ ker'}x)$$

$$M_r = \frac{K}{\ell^2}\left\{ C_1\left[-\text{bei } x - \frac{1}{x}(1-\nu)\text{ber'}x\right] + C_2\left[\text{ber } x - \frac{1}{x}(1-\nu)\text{bei'}x\right] \right.$$
$$\left. + C_3\left[\text{ker } x - \frac{1}{x}(1-\nu)\text{kei'}x\right] - C_4\left[-\text{kei } x - \frac{1}{x}(1-\nu)\text{ker'}x\right] \right\}$$

$$M_\theta = \frac{K}{\ell^2}\left\{ C_1\left[(1-\nu)\frac{1}{x}\text{ber'}x - \nu\text{ bei } x\right] + C_2\left[(1-\nu)\frac{1}{x}\text{bei'}x + \nu\text{ ber } x\right] \right. \quad (V\text{-}a)$$
$$\left. + C_3\left[(1-\nu)\frac{1}{x}\text{kei'}x + \nu\text{ ker } x\right] - C_4\left[(1-\nu)\frac{1}{x}\text{ker'}x - \nu\text{ kei } x\right] \right\}$$

$$N_r = -\frac{Et}{a}\left[ C_2 \cdot \frac{1}{x}\text{ber'}x - C_1\frac{1}{x}\text{bei'}x + C_4\frac{1}{x}\text{kei'}x \right.$$
$$\left. + C_3\frac{1}{x}\text{ker'}x + C_5\frac{1}{x^2} \right]$$

$$N_\theta = -\frac{Et}{a}\left[ C_2\left(-\text{bei } x - \frac{1}{x}\text{ber'}x\right) - C_1\left(\text{ber } x - \frac{1}{x}\text{bei'}x\right) \right.$$
$$\left. + C_4\left(\text{ker } x - \frac{1}{x}\text{kei'}x\right) + C_3\left(-\text{kei } x - \frac{1}{x}\text{ker'}x\right) - \frac{C_5}{x^2} \right]$$

$$Q_r = \frac{K}{\ell^3}\left[ -C_1\text{ bei'}x + C_2\text{ ber'}x + C_3\text{ ker'}x + C_4\text{ kei'}x \right]$$

From the above for shallow shells, the horizontal and vertical components of forces and displacements can be found with the relations:

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$$H = N_r - \frac{r}{a} Q_r$$

$$Q_v = N_r \cdot \frac{r}{a} + Q_r$$

$$\delta_h = v + \frac{r}{a} w \qquad\qquad (13)$$

$$\delta_v = \frac{r}{a} v - w$$

The solution with six constants of integration is very general, and applies to both the cases with and without a singularity at the apex.

Using the second of Eq. (13), we find

$$Q_v = -\frac{Et\ell}{a^2} \frac{c_5}{x} \qquad\qquad (V\text{-}b)$$

Flügge mentions[1] that $c_5$ is related to the total vertical force applied above a horizontal section at which the forces are considered. Thus, $c_5$ is dependent on the manner in which the spherical shell is held in vertical equilibrium. We shall discuss the cases with and without singularity and their relation to our problem separately.

a)   The Case with Singularity.     Suppose the cap in Fig. 4 is supported by a concentrated force  P  at the apex.  Then apparently  $C_5$ is equal to a certain multiple of that force.
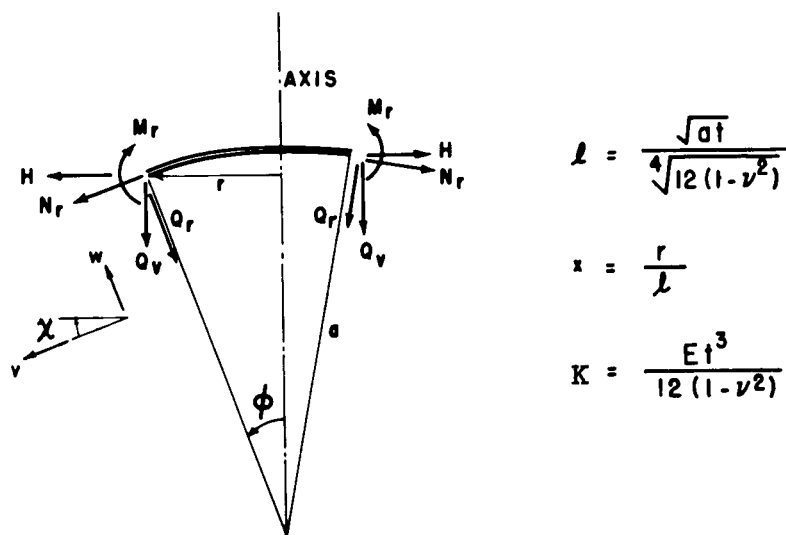
$$C_5 = - \frac{P}{2\pi} \frac{\ell^2}{K}$$



FIG. 3 - SHALLOW  SPHERICAL  CAP

$$\ell = \frac{\sqrt{at}}{\sqrt[4]{12(1-\nu^2)}}$$

$$x = \frac{r}{\ell}$$

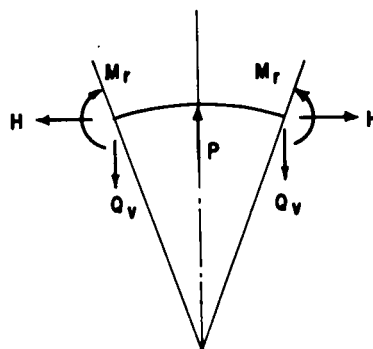$$K = \frac{Et^3}{12(1-\nu^2)}$$



FIG. 4

Since the cap is under the combined action of $P$ at the apex and $Q_v$ around the periphery, the corresponding deformation must be the change of height of the cap. If we choose the apex of the cap as reference point, i.e., $(w)_{r=0} = 0$, then $(\delta_v)_{edge} = (\delta_v)_{edge} - (w)_{r=0}$ represents the increase in height of the cap. The horizontal deformation $\delta_h$ and change of slope $\chi$ offer no ambiguity. The conditions

at $\quad r = x = 0, \quad$ $w$ and $\chi$ must be finite

at $\quad r = x = 0, \quad w = 0$

at $\quad r = x = 0, \quad N_r$ and $N_\theta$ must be finite[1]

result in $\quad C_4 = C_6 = 0, \qquad C_3 = C_5$ .

The remaining three constants $C_1$ , $C_2$ , $C_5$ can be determined by defining three quantities at the edge:

at $\quad x = x_0 \quad (r = r_0)$ , $\quad M_r = M_0$ , $\quad H = H_0$ , $\quad Q_v = Q_0$

Then the deformations of the edge can be expressed in the following forms.

$$\chi_0 = \frac{1}{\ell}(C_{11} ber'x_0 + C_{21} bei'x_0)M_0 + \frac{1}{\ell}(C_{12} ber'x_0 + C_{22} bei'x_0)H_0$$
$$+ (\frac{1}{\ell}C_{13}\,ber'x_0 + \frac{1}{\ell}C_{23}\,bei'x_0 - \frac{\ell\,r_0}{K}\,kei'x_0)\,Q_0$$

$$\delta h_0 = \frac{\ell}{a}\left\{ C_{11}\left[x_0\,ber\,x_0 - (1+\nu)\,bei'x_0\right] \right.$$
$$\left. + C_{21}\left[x_0\,bei\,x_0 + (1+\nu)\,ber'x_0\right]\right\}M_0$$
$$+ \frac{\ell}{a}\left\{ C_{12}\left[x_0\,ber\,x_0 - (1+\nu)\,bei'x_0\right] \right.$$
$$\left. + C_{22}\left[x_0\,bei\,x_0 + (1+\nu)\,ber'x_0\right]\right\}H_0$$
$$+ \left\{ \frac{\ell}{a}C_{13}\left[x_0\,ber\,x_0 - (1+\nu)\,bei'x_0\right] \right.$$
$$+ \frac{\ell}{a}C_{23}\left[x_0\,bei\,x_0 + (1+\nu)\,ber'x_0\right]$$
$$\left. - \frac{\ell^3 r_0}{a\,K}\left[x_0\,kei\,x_0 + (1+\nu)\,ker'x_0 + \frac{(1+\nu)}{x_0}\right]\right\}Q_0$$

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$$\delta v_o = \left\{ C_{11} \left[ -\frac{\ell^2}{a^2} x_o (1 + v) \text{ bei'}x_o - \text{ber } x_o \right] \right.$$

$$\left. + C_{21} \left[ \frac{\ell^2 x_o}{a^2} (1 + v) \text{ ber'}x_o - \text{bei } x_o \right] \right\} M_o$$

$$+ \left\{ C_{12} \left[ -\frac{\ell^2}{a^2} x_o (1 + v) \text{ bei'}x_o - \text{ber } x_o \right] \right.$$

$$\left. + C_{22} \left[ \frac{\ell^2 x_o}{a^2} (1 + v) \text{ ber'}x_o - \text{bei } x_o \right] \right\} H_o$$

$$+ \left\{ C_{13} \left[ -\frac{\ell^2}{a^2} x_o (1 + v) \text{ bei'}x_o - \text{ber } x_o \right] \right.$$

$$+ C_{23} \left[ \frac{\ell^2 x_o}{a^2} (1 + v) \text{ ber'}x_o - \text{bei } x_o \right]$$

$$\left. - \frac{\ell^2 r_o}{K} \left[ \frac{\ell^2 x_o}{a^2} (1 + v) (\text{ker'}x_o + \frac{1}{x_o}) - \text{kei } x_o \right] \right\} Q_o$$

where $C_{11} = -\frac{1}{\Delta} \frac{\ell^2}{K} \text{ ber'}x_o$

(VI-a)

$$C_{12} = -\frac{1}{\Delta} \frac{a^3 x_o}{Et(a^2 + r_o^2)} \left[ \text{ber } x_o - \frac{(1-v)}{x_o} \text{ bei'}x_o \right]$$

$$C_{13} = \frac{1}{\Delta} \left\{ \left[ \text{ber } x_o \cdot \text{ker'}x_o - \text{ber'}x_o \cdot \text{ker } x_o - \frac{(1-v)}{x_o} \text{ bei'}x_o \cdot \text{ker'}x_o \right. \right.$$

$$\left. + \frac{(1-v)}{x_o} \text{ ber'}x_o \cdot \text{ker'}x_o \right] \cdot \frac{\ell^2 r_o}{K} + \left[ \text{ber } x_o - \frac{(1-v)}{x_o} \text{ bei'}x_o \right] \cdot$$

$$\left. \cdot \frac{\ell^3 a^2}{K(a^2 + r_o^2)} \right\}$$

$$C_{21} = -\frac{1}{\Delta} \frac{\ell^2}{K} \text{ bei'}x_o$$

$$C_{22} = -\frac{1}{\Delta} \frac{a^3 x_o}{Et(a^2 + r_o^2)} \left[ \text{bei } x_o + \frac{(1-v)}{x_o} \text{ ber'}x_o \right]$$

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$$C_{23} = \frac{1}{\Delta} \left\{ \left[ \text{bei } x_o \cdot \text{ker'} x_o - \text{bei'} x_o \cdot \text{ker } x_o + \frac{(1-\nu)}{x_o} \text{ber'} x_o \cdot \text{ker'} x_o \right. \right.$$

$$\left. + \frac{(1-\nu)}{x_o} \text{bei'} x_o \cdot \text{kei'} x_o \right] \cdot \frac{\ell^2 r_o}{K} + \left[ \text{bei } x_o + \frac{(1-\nu)}{x_o} \text{ber'} x_o \right] \cdot$$

$$\left. \cdot \frac{\ell^3 a^2}{K(a^2 + r_o^2)} \right\}$$

and

$$\Delta = \left| \begin{matrix} -\left[ \text{bei } x_o + \dfrac{(1-\nu)}{x_o} \text{ber'} x_o \right] & \left[ \text{ber } x_o - \dfrac{(1-\nu)}{x_o} \text{bei'} x_o \right] \\ \text{bei'} x_o & -\text{ber'} x_o \end{matrix} \right|$$

From Eq. (VI-a), the following relationship can be developed.

$$\begin{Bmatrix} \chi_o \\ \delta h_o \\ \delta v_o \end{Bmatrix} = \begin{bmatrix} f_o \end{bmatrix} \cdot \begin{Bmatrix} M_o \\ H_o \\ Q_o \end{Bmatrix} \tag{14}$$

or in matrix notation: $v_o = f_o \cdot S_o$

The 3x3 flexibility matrix $f_o$ defined by (14) theoretically should be symmetrical. An actual example showed that there are slight discrepancies between terms on opposite sides of the main diagonal. This is due to the asymptotic nature of the solution and the use of 1 and $r/a$ for cosine and sine of the extended angle $\phi_o$. The use of this $f$ matrix in an actual problem in general yielded satisfactory results except for a local "hump" near the singular support. However, such local phenomenon can be minimized if the size of the cap is made very small. In certain cases, such as the examples using approximate joint loads (see examples in Section IV), a 3x3 flexibility matrix like this one is essential for obtaining the correct answer.

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b) <u>The Case Without Singularity.</u>     When the vertical force $Q_v$ is balanced by some distributed pressure instead of a single force $P$ at apex, different constants of integration are required to achieve a solution.  For finite deformations and finite membrane forces at the apex, $C_3$ and $C_4$ must vanish.  If we again consider the apex as the vertical reference point, i.e, $w = 0$ at $r = x = 0$, then $C_6 = -C_1$ . The remaining three constants $C_1$ , $C_2$ , $C_5$ are eliminated by defining three quantities $M_o$ , $H_o$ , $Q_o$ at $x = x_o$ as before.  On this basis, the expressions for edge deformations become:

$$\chi_o = -\frac{\ell}{\Delta K}\left[(ber'x_o)^2 + (bei'x_o)^2\right] \cdot M_o - \frac{a^3 x_o}{\Delta Et\,\ell(a^2 + r_o^2)} \cdot$$

$$\left[bei\,x_o \cdot bei'x_o + ber\,x_o \cdot ber'x_o\right] \cdot H_o$$

$$+ \frac{a^4}{\Delta Et\,\ell^2(a^2 + r_o^2)}\left[ber\,x_o \cdot ber'x_o + bei\,x_o \cdot bei'x_o\right] \cdot Q_o$$

$$\delta h_o = -\frac{\ell^3 x_o}{a\,K\cdot\Delta}\left\{ber\,x_o \cdot ber'x_o + bei\,x_o \cdot bei'x_o\right\} \cdot M_o$$

$$- \frac{\ell a^2}{\Delta\cdot Et(a^2 + r_o^2)}\left\{x_o^2\left[(ber\,x_o)^2 + (bei\,x_o)^2\right]\right.$$

$$+ 2\,x_o\left[bei\,x_o \cdot ber'x_o - ber\,x_o \cdot bei'x_o\right]$$

$$\left. + (1 - \nu^2)\left[(ber'x_o)^2 + (bei'x_o)^2\right]\right\} \cdot H_o$$

$$+ \frac{a^3}{\Delta\cdot Et(a^2 + r_o^2)}\left\{x_o\left[(ber\,x_o)^2 + (bei\,x_o)^2\right]\right.$$

$$+ 2\left[bei\,x_o \cdot ber'x_o - ber\,x_o \cdot bei'x_o\right]$$

$$\left. + \frac{(1-\nu^2)}{x_o}\left[(ber'x_o)^2 + (bei'x_o)^2\right]\right\} \cdot Q_o - (1 + \nu)\frac{a}{Et}\,Q_o$$

(VI-b)

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$$\delta v_o = \frac{\ell^2}{\Delta K} \left\{ \text{ber } x_o \cdot \text{ber'} x_o + \text{bei } x_o \cdot \text{bei'} x_o - (\frac{x_o^2 \ell^2}{a^2} + 1) \text{ ber'} x_o \right\} \cdot M_o$$

$$+ \frac{a^3 x_o}{\Delta \text{ Et}(a^2 + r_o^2)} \left\{ \left[ (\text{ber } x_o)^2 + (\text{bei } x_o)^2 \right] \right.$$

$$+ \left[ \frac{x_o \ell^2}{a^2} (1 + \nu) - \frac{(1 - \nu)}{x_o} \right] \cdot \left[ \text{ber } x_o \cdot \text{bei'} x_o - \text{bei } x_o \cdot \text{ber'} x_o \right]$$

$$- \frac{\ell^2}{a^2} (1 - \nu^2) \left[ (\text{ber'} x_o)^2 + (\text{bei'} x_o)^2 \right]$$

$$\left. - (\frac{x_o^2 \ell^2}{a^2} + 1) \left[ \text{ber } x_o - \frac{(1 - \nu)}{x_o} \text{ bei'} x_o \right] \right\} \cdot H_o$$

$$+ \frac{a^4}{\Delta \text{ Et} \cdot \ell (a^2 + r_o^2)} \left\{ - \left[ (\text{ber } x_o)^2 + (\text{bei } x_o)^2 \right] \right.$$

$$- \left[ \frac{x_o \ell^2}{a^2} (1 + \nu) - \frac{(1 - \nu)}{x_o} \right] \cdot \left[ \text{ber } x_o \cdot \text{bei'} x_o - \text{bei } x_o \cdot \text{ber'} x_o \right]$$

$$+ \frac{\ell^2}{a^2} (1 - \nu^2) \left[ (\text{ber'} x_o)^2 + (\text{bei'} x_o)^2 \right]$$

$$\left. + (\frac{x_o^2 \ell^2}{a^2} + 1) \left[ \text{ber } x_o - \frac{(1 - \nu)}{x_o} \text{ bei'} x_o \right] \right\} \cdot Q_o - \frac{(1 + \nu) x_o \ell}{\text{Et}} Q_o$$

If we form a 3x3 flexibility matrix $f_o$ from (VI-b) so as to satisfy (14), we find that the third column and the third row are not symmetrical with respect to the main diagonal. In this case Betti's law is not satisfied because the part of the work done by the distributed pressure in vertical direction is not included by considering only the quantities appearing in Eq. (VI-b). Such a flexibility matrix cannot be used.

However, the relationships expressed by (VI-b) are valid, and the first 2x2 matrix obtained from (VI-b) is the same as that obtained from

(VI-a). This means that the force-deformation relationships in the horizontal and rotational senses are independent of the manner in which the cap is supported vertically. In our solution process using the exact joint loads (see Sections III,C,2 and IV,A), we shall make use of such a 2x2 flexibility matrix along with the first two coefficients of the last equation of (VI-b).

## C. Joint Loads

In the finite element analysis, only the forces and displacements of certain discrete points of the structure are considered. These points are called the nodal points of the structure which are the junctions of the individual elements. For axisymmetric shells of revolution, these nodal points are circles. If the structure is subjected to concentrated forces applied at the nodal points, the results of the analysis will be exact if the geometry of the element is exact. If distributed load exists, we can either consider the distributed load as concentrated at several nodal points, thus obtaining an approximate solution, or we can hold the nodal points fixed in space and determine the fixed-edge forces of each element. The reactions of these fixed-edge forces can be applied at the nodal points as concentrated joint loads. The element forces and deformations due to these concentrated joint loads superimposed on the fixed-edge forces and deformations yield the final answers. Next, we shall discuss the two alternative procedures in more details.

1. <u>Approximate Joint Loads.</u>
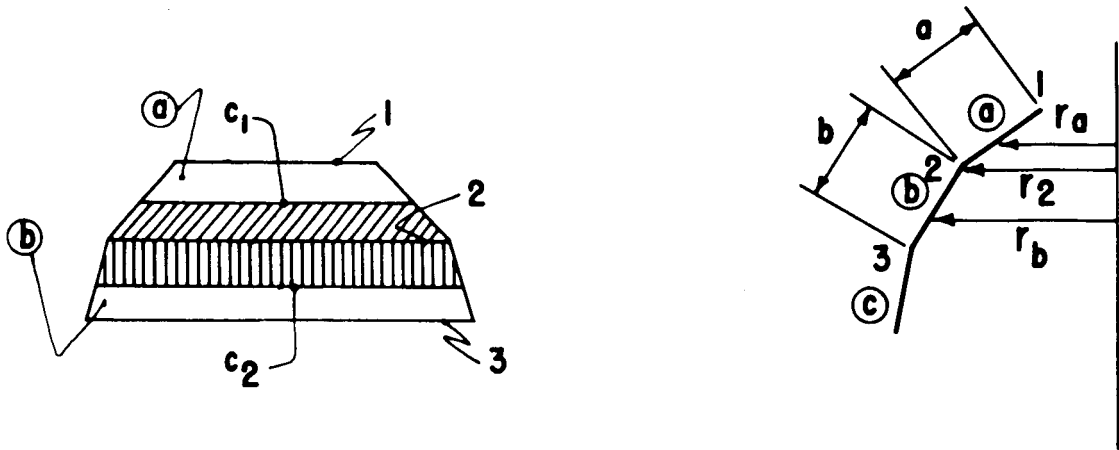


# FIG. 5

Fig. 5 shows the developed surfaces of conical elements (a) and (b) with joints 1, 2, and 3; $C_1$ and $C_2$ are their respective centroids. The approximate load on joint 2 is due to the components of the total pressure on the shaded area. Thus, we developed an expression for the joint loads due to internal pressure $p_r$.

$$T_2 = 0$$

$$P_{h2} = \left[ \left( \frac{r_a + r_2}{2r_2} \right) a \sin \alpha_a + \left( \frac{r_2 + r_b}{2r_2} \right) b \sin \alpha_b \right] p_r$$

$$P_{v2} = \left[ \left( \frac{r_a + r_2}{2r_2} \right) a \cos \alpha_a + \left( \frac{r_2 + r_b}{2r_2} \right) b \cos \alpha_b \right] p_r$$

(15)

where $T_2$, $P_{h2}$, $P_{v2}$ are the moment, horizontal and vertical forces applied at joint 2, a and b are the meridianal lengths of the elements (a) and (b),

$r_a$, $r_b$ are the horizontal radii of the centroidal
circle of elements (a) and (b)

$r_2$ = horizontal radius of joint 2

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$\alpha_a$ , $\alpha_b$ are the inclinations of elements (a) and (b) as defined in Fig. 1.

The loads contributed by pressure on the spherical cap are assumed entirely concentrated on the bottom edge of the cap.

Such approximations yield only vertical and horizontal forces and no moment for the joint loads. Actual calculations showed that the concentrated forces induce high negative moments at intermediate joints, even far from end restraints, where the action is known to be predominantly membrane.

As the sizes of the elements are made smaller, these negative joint moments decrease as expected. But in order to reduce these moments to the magnitudes of what they should be in a nearly membrane state of stress, we must divide the actual shell into a very large number of elements.

A solution using these approximate joint loads, which are considered as equivalent system to the actual loading, is quite satisfactory for the solution of problems. In this solution, it is necessary to use a 3x3 flexibility for the spherical cap (as discussed in III, B, 3-a). Otherwise, the results would show zero vertical element force in the cap which is not correct.

2. **Joint Loads from Fixed-Edge Forces of the Element.** A solution
using this type of joint loads consists of the following steps: (Fig. 6)

(1) Fix all joints in space and compute the fixed-edge
forces in all elements.

(2) Reverse these fixed-edge forces, apply them on the joints,
and let the structure deform. Compute joint displacements.
Then determine the forces and deformations in the elements.

(3) The final states of forces and deformations in the
elements are obtained by adding (1) to (2).



FIND FEF IN
ELEMENTS.

JOINT LOAD AS
REVERSED FEF

FIND JOINT DIS-
PLACEMENTS,
ELEMENT FORCES,
AND DEFORMATIONS.

( I ) + ( 2 ) GIVE
FINAL FORCES
AND DEFORMATIONS
IN ELEMENTS.

( I )

( 2 )

( 3 )
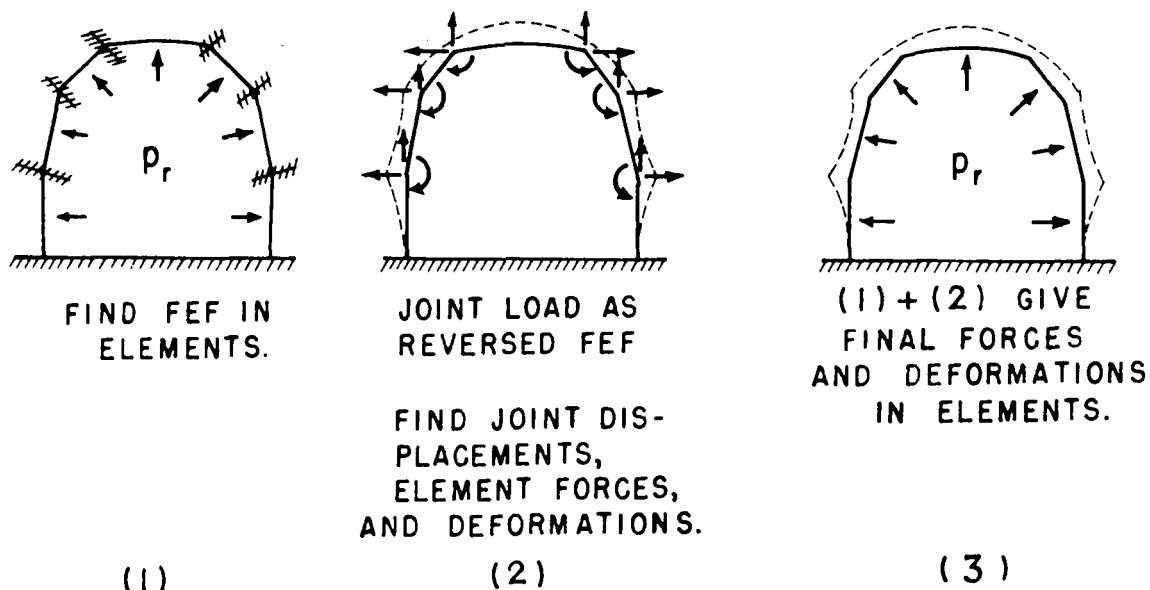
## FIG. 6

For the case of ordinary structural frames the solution is exact. For
the case of a shell, this solution, although exact in concept, does not
yield more accurate results than that found using approximate joint
loads. Moreover, this procedure involves more steps than the solution
using approximate joint loads. Therefore, it was not found to be
particularly advantageous. This will be discussed later in Section V.

The calculation of the fixed-edge forces in the spherical and
conical elements needs some further comments.

a) <u>Spherical Cap.</u>     Membrane solution of shallow spherical shell gives

$$\chi_{mo} = \frac{(1-\nu)}{2\ Et}\ p_r \cdot r_o$$

$$\delta h_{mo} = \frac{(1-\nu)}{2\ Et}\ p_r \cdot r_o \cdot a \tag{16}$$

$$\delta v_{mo} = \frac{(1-\nu)}{2\ Et}\ p_r \cdot r_o^{\ 2}$$

Eq. (16) gives the membrane deformations $\{v_{mo}\}$ which have been defined in Fig. 3. (Subscript $m$ indicates membrane action). The application of a set of edge forces $\{S_{F_o}\}$ will produce edge deformations $\{v_{F_o}\}$ which when added to $\{v_m\}$ result in zero deformations. Thus,

$$v_{m_o} + v_{F_o} = 0$$

Since

$$v_{F_o} = f_o \cdot S_{F_o}$$

$$f_o \cdot S_{F_o} = -v_{m_o} \tag{17}$$

$$S_{F_o} = -f_o^{-1} \cdot v_{m_o}$$

For the case without singularity at apex, we use a 2x2 $f_o$ matrix (refer to III, B, 3). Hence

$$\begin{Bmatrix} M_F \\ H_F \end{Bmatrix} = -\begin{bmatrix} k_o \end{bmatrix} \cdot \begin{Bmatrix} \chi_{mo} \\ \delta h_{mo} \end{Bmatrix} \tag{17a}$$

The third component of the element forces can be found separately:

$$Q_{vF} = \tfrac{1}{2}\ p_r \cdot a \cdot \sin \phi_o \tag{18}$$

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Fig. 7 shows the relations between the fixed edge forces of the cap
and the joint forces, from which it is apparent that
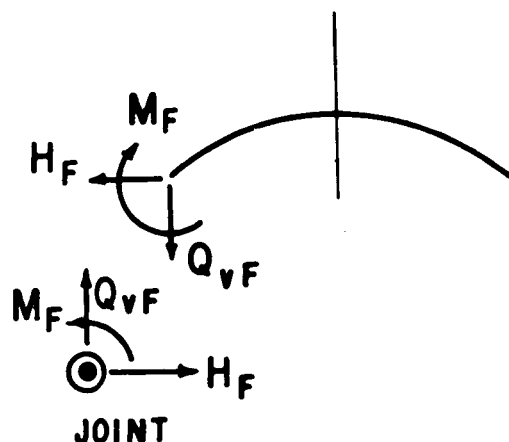


$$T = -M_F$$

$$P_h = -H_F$$

$$P_v = Q_{vF}$$

$$(19)$$

or

$$\{R\} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \{S_F\}$$

**FIG. 7**

The 3x3 matrix in (19) is the equilibrium matrix of the spherical cap
$A_o$ (see III, D, 1) with a minus sign. Hence

$$R = -A_o \cdot S_F \qquad (20)$$

b) <u>Conical Ring.</u>
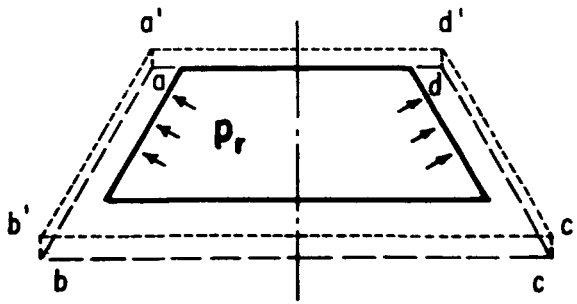


**FIG. 8**

Fig. 8 shows a conical ring under internal pressure $p_r$. To find the membrane displacements, we need some arbitrary support to balance the pressure vertically. For this purpose we choose to hold the cone at upper edge such that the meridianal displacement v of the top edge is zero. This, we recall, is the same boundary condition as used in determining the meridianal stretch in the flexibility matrix (see III, B, 1).

Under this assumption the displaced cone will be in the dotted position a'b'c'd'. (Note that angle changes are not shown in Fig. 8.) A rigid body displacement will bring the cone to the dashed position a b c d where the top edge of the cone is at the original level. We see that such a movement does not change the magnitudes of the horizontal displacements $\delta_{mi}$ and $\delta_{mj}$ and the meridianal stretch $e_m$. Thus the stiffness matrix obtained in III, B, 1 can be used directly to find the fixed-edge forces.

Membrane solution of conical shells gives:

$$N_{mi} = \frac{p_r \cot \alpha}{2} (s_i - \frac{s_j^2}{s_i})$$

$$\bar{\chi}_{mi} = - \frac{p_r \cot^2 \alpha}{2 Et} (3 s_i + \frac{s_j^2}{s_i})$$

$$\bar{\chi}_{mj} = \frac{p_r \cot^2 \alpha}{Et} \cdot 2 s_j \qquad (21)$$

$$\bar{\delta}_{mi} = -\frac{p_r \cos^2 \alpha}{2 \, Et \, \sin \alpha} \left[ (2-\nu)s_i^2 + \nu \, s_j^2 \right]$$

$$\bar{\delta}_{mj} = \frac{p_r \cos^2 \alpha}{Et \, \sin \alpha} \cdot s_j^2$$

$$e_m = \frac{p_r \cot \alpha}{4 \, Et} \left[ (1-2\nu)(s_j^2 - s_i^2) - 2 \, s_j^2 \cdot \log \frac{s_j}{s_i} \right]$$

By the same reasoning as followed in (a)

$$S_F = -f^{-1} v_m = -k \, v_m \tag{22}$$

$$\text{where} \quad v_m = \begin{Bmatrix} \bar{\chi}_{mi} \\ \bar{\chi}_{mj} \\ \bar{\delta}_{mi} \\ \bar{\delta}_{mj} \\ e_m \end{Bmatrix} \quad \text{and} \quad S_F = \begin{Bmatrix} M_{Fi} \\ M_{Fj} \\ H_{Fi} \\ H_{Fj} \\ N_{Fj} \end{Bmatrix}$$

Also

$$R = -A \, S_F = A \, k \, v_m \tag{23}$$

or for the $n^{th}$ element in general

$$R^{(n)} = \begin{Bmatrix} R_i^{(n)} \\ R_j^{(n)} \end{Bmatrix} = A_n \, k^{(n)} \, v_m^{(n)} \tag{23-a}$$

where $A_n$ is the equilibrium matrix relating the edge forces to the joint forces of the $n^{th}$ element (see III, D, 1).

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c) <u>Cylindrical Ring.</u>     Equations (22) and (23) still hold except the stiffness and equilibrium matrices used should be those of the cylindrical element.   The membrane deformations $\{v_m\}$ in this case are

$$\bar{\chi}_{mi} = 0$$

$$\bar{\chi}_{mj} = 0$$

$$\bar{\delta}_{mi} = -\frac{p_r a^2}{Et} \tag{24}$$

$$\bar{\delta}_{mj} = \frac{p_r a^2}{Et}$$

$$e_m = -\frac{v\, p_r \cdot a \cdot \ell}{Et}$$

With the joint loads determined in the above described manner, a standard matrix solution will yield

$$\bar{S}^{(n)} = \begin{Bmatrix} \bar{M}_i^{(n)} \\ \bar{M}_j^{(n)} \\ \bar{H}_i^{(n)} \\ \bar{H}_j^{(n)} \\ \bar{N}_j^{(n)} \end{Bmatrix}$$

which represent only the effect of the system of joint loads resulting from the unbalanced fixed-edge forces.   The final edge forces in the $n^{th}$ element are

$$S^{(n)} = \bar{S}^{(n)} + S_F^{(n)} \tag{25}$$

while $N_i^{(n)} = \bar{r}^{(n)} N_j^{(n)} + N_{mi}^{(n)}$ .

## D.  Matrix Solution of the Problem

1.  _The Standard Matrix Operations._   It was mentioned at the begin-
ning of this section that four kinds of matrices are needed to solve a
problem of **axis**ymmetrical shells of revolution.  We have already discussed
the formulation and application of the stiffness and load matrices.  Now
we shall show the formulation of the displacement transformation and
equilibrium matrices.

Fig. 9 shows the positive directions of the loads and displacements
of the nodal points or joints of the structure which are adopted in this
report.
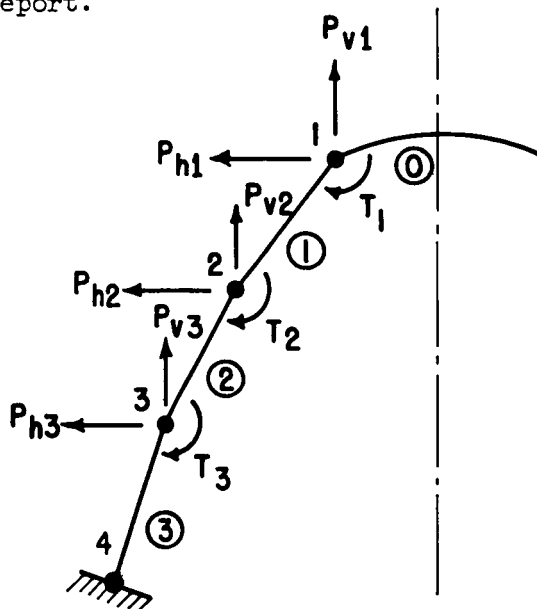


Joint displacements:

$\theta$  -  in direction of  T,
positive when flattens
the element.

$\Delta_h$  -  in direction of  $P_h$,
positive outward.

$\Delta_v$  -  in direction of  $P_v$,
positive upward.

ELEMENTS = ⓪,①,②, ... ⓝ
JOINTS   = 1, 2, 3, ... n+1

## FIG. 9 - JOINT FORCES AND DISPLACEMENTS.

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Remembering the definitions of the element forces and deformations (Figs. 1 and 2) we obtain the following relations:

$$
\left\{
\begin{aligned}
\chi_o &= \theta_1 \\
\delta h_o &= \Delta_{h1} \\
\delta v_o &= -\Delta_{v1}
\end{aligned}
\right.
$$

$$
\left\{
\begin{aligned}
\bar{\chi}_i^{(1)} &= -\theta_1 \\
\bar{\chi}_j^{(1)} &= +\theta_2 \\
\bar{\delta}_i^{(1)} &= -\Delta_{h1} \\
\bar{\delta}_j^{(1)} &= +\Delta_{h2} \\
e^{(1)} &= -\cos\alpha_1 \cdot \Delta_{h1} + \sin\alpha_1 \cdot \Delta_{v1} + \cos\alpha_1 \Delta_{h2} - \sin\alpha_1 \cdot \Delta_{v2}
\end{aligned}
\right.
$$

$$
\left\{
\begin{aligned}
\bar{\chi}_i^{(n)} &= -\theta_n \\
\bar{\chi}_j^{(n)} &= +\theta_{n+1} \\
\bar{\delta}_i^{(n)} &= -\Delta_{hn} \\
\bar{\delta}_j^{(n)} &= +\Delta_{hn+1} \\
e^{(n)} &= -\cos\alpha_n \cdot \Delta_{hn} + \sin\alpha_n \Delta_{vn} + \cos\alpha_n \cdot \Delta_{hn+1} - \sin\alpha_n \cdot \Delta_{vn+1}
\end{aligned}
\right.
$$

In matrix form:

$$
\begin{Bmatrix}
\varkappa_o \\
\delta h_o \\
\delta v_o \\
\bar{\varkappa}_i^{(1)} \\
\bar{\varkappa}_j^{(1)} \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\bar{\delta}_i^{(n)} \\
\bar{\delta}_j^{(n)} \\
e^{(n)}
\end{Bmatrix}
=
\begin{bmatrix}
1 & 0 & 0 & & & & & & \\
0 & 1 & 0 & & & & & & \\
0 & 0 & -1 & & & & & & \\
-1 & 0 & 0 & 0 & & & & & \\
0 & 0 & 0 & 1 & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & -1 & 0 & 0 & 0 & 0 \\
& & & & & 0 & 0 & 0 & 1 & 0 \\
& & & & & -\cos\alpha_n & \sin\alpha_n & 0 & \cos\alpha_n & -\sin\alpha_n
\end{bmatrix}
\begin{Bmatrix}
\theta_1 \\
\Delta_{h1} \\
\Delta_{v1} \\
\theta_2 \\
\Delta_{h2} \\
\Delta_{v2} \\
\cdot \\
\cdot \\
\theta_n \\
\Delta_{hn} \\
\Delta_{vn} \\
\theta_{n+1} \\
\Delta_{hn+1} \\
\Delta_{vn+1}
\end{Bmatrix}
$$

or

$$
\begin{Bmatrix}
v^{(0)} \\
v^{(1)} \\
v^{(2)} \\
\cdot \\
\cdot \\
v^{(n)}
\end{Bmatrix}
=
\begin{bmatrix}
\begin{smallmatrix}B_0 \\ 3\times3\end{smallmatrix} & & & \\
& \begin{smallmatrix}B_1 \\ 5\times6\end{smallmatrix} & & \\
& & \begin{smallmatrix}B_2 \\ 5\times6\end{smallmatrix} & \\
& & & \begin{smallmatrix}B_n \\ 5\times6\end{smallmatrix}
\end{bmatrix}
\cdot
\begin{Bmatrix}
r_1 \\
r_2 \\
r_3 \\
\cdot \\
\cdot \\
r_{n+1}
\end{Bmatrix}
$$

Using symbolic notations:

$$v = B r \qquad\qquad (26)$$

where $v$ is the matrix representing edge deformations of all elements

$r$ is the matrix representing the displacements of all joints

and $B$ is the displacement transformation matrix.

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Next, considering the equilibrium of each joint, with proper attention to the positive directions of the joint forces and element forces as defined in Fig. 1,2, and 9, we have

$$
\left\{
\begin{aligned}
T_1 &= M_o - M_i^{(1)} \\
P_{h1} &= H_o - H_i^{(1)} - \overline{r}^{(1)} \cos \alpha_1 \, N_j^{(1)} \\
P_{v1} &= -Qv_o + \overline{r}^{(1)} \sin \alpha_1 \cdot N_j^{(1)}
\end{aligned}
\right.
$$

$$
\left\{
\begin{aligned}
T_2 &= M_j^{(1)} - M_i^{(2)} \\
P_{h2} &= H_j^{(1)} - H_i^{(2)} + \cos \alpha_1 \cdot N_j^{(1)} - \overline{r}^{(2)} \cos \alpha_2 \, N_j^{(2)} \\
P_{v2} &= -\sin \alpha_1 \cdot N_j^{(1)} + \overline{r}^{(2)} \sin \alpha_2 \cdot N_j^{(2)}
\end{aligned}
\right.
$$

$$
\left\{
\begin{aligned}
T_n &= M_j^{(n-1)} - M_i^{(n)} \\
P_{hn} &= H_j^{(n-1)} - H_i^{(n)} + \cos \alpha_{n-1} \cdot N_j^{(n-1)} - \overline{r}^{(n)} \cos \alpha_n \cdot N_j^{(n)} \\
P_{vn} &= -\sin \alpha_{n-1} \cdot N_j^{(n-1)} + \overline{r}^{(n)} \sin \alpha_n \cdot N_j^{(n)}
\end{aligned}
\right.
$$

$$
\left\{
\begin{aligned}
T_{n+1} &= M_j^{(n)} \\
P_{hn+1} &= H_j^{(n)} + \cos \alpha_n \cdot N_j^{(n)} \\
P_{vn+1} &= -\sin \alpha_n \cdot N_j^{(n)}
\end{aligned}
\right.
$$

Note: for cylindrical element, $\alpha = 90^{\circ}$ $\sin \alpha = 1$, $\cos \alpha = 0$, $\overline{r} = 1$, the above relations still hold.

From these relations we can form

$$
\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ R_{n+1} \end{Bmatrix}
\quad
\begin{bmatrix} \underset{3\times3}{A_o} & \underset{3\times5}{A_1} & & & & \\ & \underset{3\times10}{A_2} & & & \\ & & \underset{3\times10}{A_3} & & \\ & & & & \\ & & & \underset{3\times10}{A_{n-1}} & \\ & & & & \underset{3\times5}{A_n} \end{bmatrix}
\cdot
\begin{Bmatrix} S^{(0)} \\ S^{(1)} \\ S^{(2)} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ S^{(n)} \end{Bmatrix}
$$

or using symbolic notations:

$$R = A S \tag{27}$$

where   R   is the matrix representing forces on all joints

      S   is the matrix representing edge forces of all elements

and   A   is the equilibrium matrix.

With   v   and   S   as defined in Eqs. (26) and (27), we can write

$$S = k v \tag{28}$$

where

$$
k = \begin{bmatrix} k^{(0)} & & & & \\ & k^{(1)} & & & \\ & & k^{(2)} & & \\ & & & \ddots & \\ & & & & k^{(n)} \end{bmatrix}
$$

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and $k^{(0)} = f_o^{-1}$ , $k^{(1)} = f_1^{-1}$ , ............... $k^{(n)} = f_n^{-1}$ ,

$f_o$ is the flexibility matrix of the cap

$f_1$ ........ $f_n$ are the flexibility matrices of the $1^{st}$........ $n^{th}$ cone or cylinder.

Eqs. (27), (28), (26) yield

$$R = A \, k \, B \, r = K \, r$$
$$K = A \, k \, B$$

(29)

which relates directly the joint displacements $r$ to the joint force R of the structure. The quantity K is called the stiffness of the structure.

With a given set of joint loads R we can find in sequence: $r$ from (29), v from (26) and S from (28).

This process of solution, however, involves the operations with large matrices, such as A, B, k, consisting mostly of zeros and therefore, poses a problem of storage for the computer. Indeed, IBM 7090 can handle problems using only about 15 elements when this process is employed.

One way to avoid such a storage problem is to form the structural stiffness K directly by operating with small matrices $A_n$ , $B_n$ and $k^{(n)}$ instead of the large matrices A, B, and k .

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2. <u>The Direct Stiffness Method.</u>    To avoid the difficulty of having inadequate storage locations in the computer, we operate with the small submatrices $A_n$ , $B_n$ of the large matrices $A$ and $B$ appearing in Eqs. (27) and (26) together with the element stiffness $k^{(n)}$ and form the structural stiffness $K$ in a different way.

In the previous article we have established for the first conical element the following relations:

$$\begin{aligned} S^{(1)} &= k^{(1)} \quad v^{(1)} \\ v^{(1)} &= B_1 \quad r_E^{(1)} \\ R_E^{(1)} &= A_1 \quad S^{(1)} \end{aligned} \tag{30}$$

where
$$R_E^{(1)} = \left\{ \begin{array}{c} R_i^{(1)} \\ R_j^{(1)} \end{array} \right\} = \left\{ \begin{array}{c} T_i \\ P_{hi} \\ P_{vi} \\ T_j \\ P_{hj} \\ P_{vj} \end{array} \right\}^{(1)} \qquad r_E^{(1)} = \left\{ \begin{array}{c} r_i^{(1)} \\ r_j^{(1)} \end{array} \right\} = \left\{ \begin{array}{c} \theta_i \\ \Delta_{hi} \\ \Delta_{vi} \\ \theta_j \\ \Delta_{hj} \\ \Delta_{vj} \end{array} \right\}^{(1)}$$

are the joint forces and joint displacements of the structure consisting of only one element, i.e., element (1).

Thus, the stiffness of the structure consisting of only element (1) is

$$K^{(1)} = A_1 k^{(1)} B_1$$

and similarly for the $n^{th}$ structure, that is, the structure consisting of the $n^{th}$ cone (or cylinder), only

$$K^{(n)} = A_n k^{(n)} B_n \tag{31}$$

For the spherical cap

$$K^{(0)} = A_0 \, k^{(0)} \, B_0$$

and

$$R_E^{(0)} = K^{(0)} \, r_E(0)$$

where $R_E^{(0)} = R_j^{(0)}$, $r_E^{(0)} = r_j^{(0)}$, since the cap has only a lower edge.

By partitioning of matrices, Eq. (31) can be rewritten as

$$R_E^{(n)} = \begin{Bmatrix} R_i^{(n)} \\ R_j^{(n)} \end{Bmatrix} = \begin{bmatrix} \bar{k}_{ii}^{(n)} & \bar{k}_{ij}^{(n)} \\ \bar{k}_{ji}^{(n)} & \bar{k}_{jj}^{(n)} \end{bmatrix} \cdot \begin{Bmatrix} r_i^{(n)} \\ r_j^{(n)} \end{Bmatrix}$$

where
$$\bar{k}_{ii}^{(n)} = A_{ni} \cdot k^{(n)} \cdot B_{ni}$$

$$\bar{k}_{ij}^{(n)} = A_{ni} \cdot k^{(n)} \cdot B_{nj}$$

$$\bar{k}_{ji}^{(n)} = A_{nj} \cdot k^{(n)} \cdot B_{ni}$$

$$\bar{k}_{jj}^{(n)} = A_{nj} \cdot k^{(n)} \cdot B_{nj}$$

(32)

and $A_{ni}$, $A_{nj}$, $B_{ni}$, $B_{nj}$ are submatrices of $A_n$ and $B_n$ such that

$$\begin{bmatrix} A_n \end{bmatrix} = \begin{bmatrix} A_{ni} \\ A_{nj} \end{bmatrix}, \qquad \begin{bmatrix} B_n \end{bmatrix} = \begin{bmatrix} B_{ni} & B_{nj} \end{bmatrix} \ .$$

Next, we note that the joint force at any joint in the assemblage is the sum of the forces on joints $i$ and $j$ of the lower and upper elements respectively, and the joint displacement of any joint in the assemblage is common for joints $i$ and $j$ of the lower and upper elements.

This means that

$$R_1 = R_i^{(1)} + R_j^{(0)} \qquad\qquad r_1 = r_i^{(1)} = r_j^{(0)}$$

$$R_2 = R_j^{(1)} + R_i^{(2)} \qquad\qquad r_2 = r_j^{(1)} = r_i^{(2)}$$

$$\text{------------------------} \quad\text{and}\quad \text{------------------------} \qquad (33)$$

$$R_n = R_j^{(n-1)} + R_i^{(n)} \qquad\qquad r_n = r_j^{(n-1)} + r_i^{(n)}$$

$$R_{n+1} = R_j^{(n)} \qquad\qquad\qquad r_{n+1} = r_j^{(n)}$$

Using Equations (32) and (33), the structural stiffness of the assemblage may now be put into a tri-diagonal matrix form

$$
\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ \cdot \\ \cdot \\ R_{n+1} \end{Bmatrix} =
\begin{bmatrix}
(\bar{k}^{(0)}+\bar{k}_{ii}^{(1)}) & \bar{k}_{ij}^{(1)} & \cdot & \cdot & \cdot \\
\bar{k}_{ji}^{(1)} & (\bar{k}_{jj}^{(1)}+\bar{k}_{ii}^{(2)}) & \bar{k}_{ij}^{(2)} & \cdot & \cdot \\
\cdot & \bar{k}_{ji}^{(2)} & (\bar{k}_{jj}^{(2)}+\bar{k}_{ii}^{(3)}) & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & (\bar{k}_{jj}^{(n-1)}+\bar{k}_{ii}^{(n)}) & \bar{k}_{ij}^{(n)} \\
\cdot & \cdot & \cdot & \bar{k}_{ji}^{(n)} & \bar{k}_{jj}^{(n)}
\end{bmatrix}
\cdot
\begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ \cdot \\ \cdot \\ r_{n+1} \end{Bmatrix}
\qquad (34)
$$

which is an expanded form of the relation

$$R = K \cdot r$$

and it should be noted that $K$ as developed in Eq. (29) also can be stated in this form.

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The process described in this article has the advantage that $\bar{k}_{ii}$ , $\bar{k}_{ij}$ , $\bar{k}_{ji}$ , $\bar{k}_{jj}$ for each element are developed in a do-loop of the computer program, eliminating the storage of the large matrices A, B, and k. The final K matrix is the only large matrix which remains for subsequent calculations. If this K matrix is stored in a conventional rectangular form, the 7090 computer can handle problems involving 45 conical elements. By the use of subroutines of skew storage and solution[*] or by applying a recursion process[21] problems involving several hundred or more elements can be analyzed.

---

[*]Private communication from Messrs. A. De Fries and Ashvin Shah, Graduate students of University of California, Berkeley.

### 3. Special Methods for Shells Subject to Internal Pressure.

For shells subject to internal pressure only in which the stress conditions are essentially membrane except in regions near the restraints, some short-cut procedures can be used instead of the more general approaches discussed above. Two such methods are described below.

    a) Substitution of a known element force in place of a part of the structure.



(a)          (b)

## FIG. 10

Fig. 10(a) shows a spherical shell under internal pressure $p_r$ . It is known that the portion above joint a is essentially in membrane state of stress and the membrane force $N_\phi$ at a can be readily determined. We can, therefore, consider the portion of the shell below a , apply $N_\phi$ as an additional joint load at a (Fig. 10-b) and thus, reduce the number of elementary cones which need to be considered. This procedure simplifies the solution.

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b) Membrane solution plus edge-displacement effect. Fig. 11(a) shows the spherical shell subject to internal pressure $p_r$ .



( a )          ( b )          ( c )
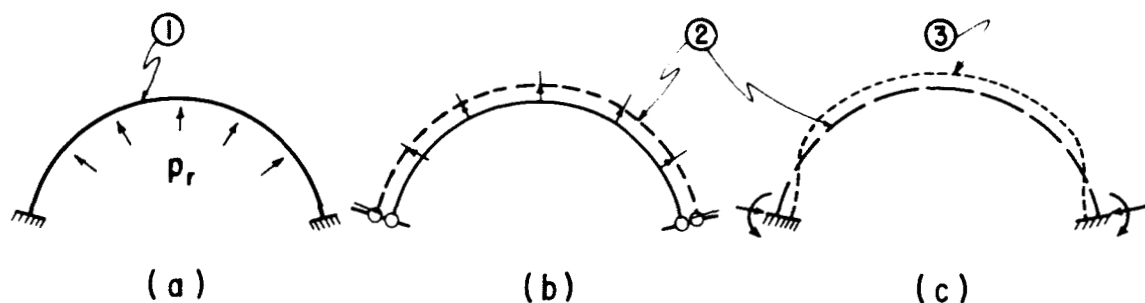
FIG. 11

We first remove the end constraints and let it deform entirely as a membrane. The shell will take the configuration shown by dashed line (2) (Figs. 11-b and 11-c). This deformed shape and the induced internal forces can be easily obtained from the membrane solution. Then we impose the known displacements at the boundary to bring the edge back to its original location and slope. At this stage bending moments and cross shears are induced in the shell, and the final configuration of the shell is shown by the dotted line (3) in Fig. 11-c. In applying this procedure, finite element analysis is used only in the last step of analysis in which only displacements at the boundary are applied. Here the problem of local moments or uneven meridianal forces around joints does not arise.

It should be remembered that the membrane solution is a convenient approximation. It represents a simplified particular solution of the governing differential equation of the shell. But for most cases, the error introduced in such an approximation is found to be negligible.

# IV. APPLICATIONS OF FINITE ELEMENT SOLUTION

## A. Examples 1-10

A large number of examples had been worked out in order to test the correctness of the formulations presented in the previous sections and to explore the advantages or disadvantages of each particular sequence of operations. A number of these examples will be presented in this chapter to illustrate applications of the method and to illustrate some of the difficulties encountered in solving problems.

For convenience of comparison, most of the examples involve the analysis of the same shell by different procedures. Thus, methods of using different joint loads, different number of elements, different stiffness of spherical cap, and special short-cut procedures are illustrated. Only one example is solved for a cylindrical pipe to illustrate the application of cylindrical elements.

The shell analyzed in most of the examples is a spherical one of constant thickness fixed at the bottom with a central semi-angle of $75^{\circ}$, a radius of 100 in. and thickness of 1/2 in., subject to an internal pressure of 100 psi. The assumed material properties are $E = 10^7$ psi and $\nu = 0.2$.

Hetényi's solution of a spherical shell[22] which is believed to be sufficiently accurate for all practical purposes is used as a basis of comparison. Rajan also formulated a solution for spherical shells[23] in closed mathematical form; the critical values of meridianal moments obtained by using this method are also listed for comparison.

Several steps are involved in solving the problem. First, decision must be made regarding the number of elements to be used to approximate the

actual shell and where the joints should be located. When the locations of the joints are selected, the geometry of the conical elements can be computed and from this the appropriate Thompson functions are obtained using certain parameters. The flexibilities and/or stiffness of the elements are then developed using these Thompson functions. Separate computer programs have been written for these purposes including the calculations of approximate joint loads. When the stiffnesses of the elements are available, one single large program takes care of all the remaining steps of the solution including the development of joint loads from fixed-edge forces if necessary. All computations are carried out by IBM 7090 computer, under the UCB Computer Center job number 255.

The criterion for judging the accuracy of a solution is a comparison with other known solutions of the distribution of the meridianal moments in general and especially of their critical values at the fixed edge. In a few cases the meridianal tension is also shown.

Ex. 1
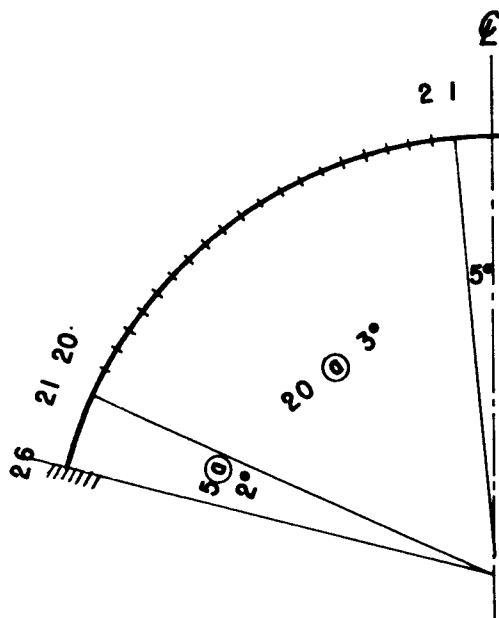


FIG. 12 - 25 CONE ARRANGEMENT
OF 75° SPHERE.

The $75°$ sphere was analyzed as an assemblage of a $5°$ spherical cap and 25 cones as shown in Fig. 12. Approximate joint loads were used.

The meridianal moments $M_s$, membrane force $N_s$ at each joint and the displacements $\Delta_h$ and $\Delta_v$ of each joint are tabulated below.

In the table, $N_s$ below joint indicates the $N_i$ force in the element below the joint, $N_s$ above joint indicates the $N_j$ force in the element above the joint.

For each element $N_i = \bar{r} \cdot N_j$.

| Joint | $\phi$ | $M_s$ | $N_s$ Above | Below | $\Delta_h$ | $\Delta_v$ | Joint | $\phi$ | $M_s$ | $N_s$ Above | Below | $\Delta_h$ | $\Delta_v$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 5° | -608 | | 6731 | .0091 | .0931 | 14 | 44 | -113 | 4862 | 5133 | .0557 | .0630 |
| 2 | 8 | -155 | 4215 | 6027 | .0135 | .0990 | 15 | 47 | -111 | 4875 | 5117 | .0588 | .0602 |
| 3 | 11 | - 85 | 4396 | 5724 | .0159 | .0882 | 16 | 50 | -112 | 4887 | 5111 | .0624 | .0578 |
| 4 | 14 | - 97 | 4514 | 5551 | .0191 | .0829 | 17 | 53 | -134 | 4905 | 5120 | .0659 | .0552 |
| 5 | 17 | -110 | 4595 | 5446 | .0230 | .0812 | 18 | 56 | -112 | 4933 | 5109 | .0671 | .0510 |
| 6 | 20 | -114 | 4655 | 5370 | .0272 | .0802 | 19 | 59 | -107 | 4941 | 5096 | .0683 | .0467 |
| 7 | 23 | -114 | 4701 | 5316 | .0312 | .0789 | 20 | 62 | -103 | 4948 | 5084 | .0706 | .0433 |
| 8 | 26 | -114 | 4738 | 5272 | .0351 | .0773 | 21 | 65 | -104 | 4955 | 5061 | .0752 | .0408 |
| 9 | 29 | -114 | 4767 | 5238 | .0388 | .0754 | 22 | 67 | -127 | 4981 | 5054 | .0761 | .0381 |
| 10 | 32 | -114 | 4792 | 5208 | .0425 | .0733 | 23 | 69 | -165 | 4984 | 5052 | .0687 | .0321 |
| 11 | 35 | -115 | 4813 | 5185 | .0460 | .0710 | 24 | 71 | -161 | 4987 | 5045 | .0495 | .0220 |
| 12 | 38 | -115 | 4832 | 5168 | .0494 | .0685 | 25 | 73 | + 15 | 4990 | 5042 | .0197 | .0091 |
| 13 | 41 | -114 | 4848 | 5149 | .0526 | .0659 | 26 | 75 | +544 | 4993 | | 0 | 0 |

NOTE:  All quantities are in inch and pound units.

Ex. 2  The same assemblage as in Ex. 1 was analyzed with joint loads derived from the fixed-edge forces of the elements. A 2 x 2 stiffness was used for the cap. The results are tabulated below.

| Joint | $M_s$ | $N_s$ Above | $N_s$ Below | $\Delta_h$ | $\Delta_v$ | Joint | $M_s$ | $N_s$ Above | $N_s$ Below | $\Delta_h$ | $\Delta_v$ |
|-------|-------|-------|-------|------|------|-------|-------|-------|-------|------|------|
| 1 | -507 | | 3849 | .0132 | .1143 | 14 | 110 | 5141 | 4870 | .0552 | .0629 |
| 2 | - 36 | 6147 | 4216 | .0148 | .1029 | 15 | 111 | 5127 | 4882 | .0581 | .0599 |
| 3 | 68 | 5780 | 4408 | .0165 | .0898 | 16 | 111 | 5114 | 4894 | .0608 | .0567 |
| 4 | 81 | 5589 | 4526 | .0195 | .0841 | 17 | 113 | 5102 | 4905 | .0633 | .0533 |
| 5 | 85 | 5470 | 4607 | .0232 | .0821 | 18 | 116 | 5092 | 4915 | .0658 | .0499 |
| 6 | 91 | 5389 | 4666 | .0272 | .0808 | 19 | 119 | 5082 | 4924 | .0688 | .0469 |
| 7 | 97 | 5330 | 4711 | .0311 | .0793 | 20 | 110 | 5072 | 4933 | .0729 | .0444 |
| 8 | 101 | 5285 | 4747 | .0349 | .0775 | 21 | 59 | 5063 | 4960 | .0754 | .0410 |
| 9 | 103 | 5250 | 4776 | .0386 | .0755 | 22 | -1.2 | 5038 | 4964 | .0739 | .0373 |
| 10 | 105 | 5220 | 4800 | .0421 | .0733 | 23 | - 60 | 5034 | 4967 | .0669 | .0315 |
| 11 | 107 | 5196 | 4821 | .0456 | .0709 | 24 | - 64 | 5031 | 4971 | .0487 | .0217 |
| 12 | 108 | 5175 | 4839 | .0489 | .0684 | 24 | 112 | 5027 | 4974 | .0195 | .0090 |
| 13 | 109 | 5157 | 4855 | .0522 | .0657 | 26 | 642 | 5024 | | 0 | 0 |

Ex. 3 The same assemblage as in Ex. 1 was permitted to deflect as a membrane, then the edge was forced back to its original position and inclination.

| Joint | $M_s$ | $N_s$ Above | $N_s$ Below | $\Delta_h$ | $\Delta_v$ | Joint | $M_s$ | $N_s$ Above | $N_s$ Below | $\Delta_h$ | $\Delta_v$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | -7.2 | | -53.0 | .0070 | .0841 | 14 | -0.3 | -1.1 | -1.1 | .0556 | .0630 |
| 2 | -3.5 | -33.2 | -22.8 | .0112 | .0846 | 15 | -.13 | -1.0 | - .9 | .0585 | .0600 |
| 3 | -0.9 | -16.6 | -12.7 | .0153 | .0840 | 16 | -.27 | - .9 | - .8 | .0613 | .0569 |
| 4 | -0.1 | -10.0 | - 8.1 | .0194 | .0831 | 17 | -.02 | - .8 | - .8 | .0638 | .0535 |
| 5 | -.05 | - 6.7 | - 5.6 | .0234 | .0819 | 18 | +1.6 | - .8 | - .7 | .0662 | .0501 |
| 6 | -.07 | - 4.8 | - 4.1 | .0274 | .0806 | 19 | +4.8 | - .7 | - .7 | .0688 | .0468 |
| 7 | -.08 | - 3.6 | - 3.3 | .0313 | .0791 | 20 | +3.3 | - .7 | - .6 | .0722 | .0439 |
| 8 | -.06 | - 2.9 | - 2.5 | .0351 | .0773 | 21 | -22.8 | - .6 | - .6 | .0757 | .0410 |
| 9 | -.05 | - 2.3 | - 2.1 | .0388 | .0754 | 22 | -64.2 | - .6 | - .6 | .0753 | .0378 |
| 10 | -.04 | - 1.9 | - 1.7 | .0424 | .0726 | 23 | -112.8 | - .6 | - .6 | .0681 | .0318 |
| 11 | -.03 | - 1.6 | - 1.5 | .0459 | .0710 | 24 | -112.2 | - .6 | - .6 | .0492 | .0219 |
| 12 | -.02 | - 1.4 | - 1.3 | .0493 | .0685 | 25 | +63.9 | - .6 | - .6 | .0197 | .0092 |
| 13 | -.01 | - 1.2 | - 1.2 | .0525 | .0658 | 26 | +595.4 | - .6 | | 0 | 0 |

Note: 1. Add 500 #/in (the constant membrane force) to the results of bending analysis to obtain the total $N_s$ in the shell.

2. The displacement values listed are total displacements caused by the membrane and bending displacements.

It is significant to list for the purposes of comparison the $M_s$ values for the same shell based on Hetényi's and Rajan's solutions. The accuracy of these solutions is known to be very good.

## Hetenyi's Solution

| $\phi$ | $M_s$ | $\phi$ | $M_s$ | $\phi$ | $M_s$ | $\phi$ | $M_s$ |
|---|---|---|---|---|---|---|---|
| 5° | | 26 | -.00012 | 47 | -.1101 | 67 | -63.66 |
| 8 | | 29 | -.00044 | 50 | -.2491 | 69 | -112.1 |
| 11 | Less than | 32 | -.00051 | 53 | -.0042 | 71 | -111.7 |
| 14 | | 35 | +.00132 | 56 | +1.636 | 73 | +62.44 |
| 17 | $10^{-4}$ | 38 | +.00706 | 59 | +4.839 | 75 | +589.2 |
| 20 | | 41 | +.0119 | 62 | +3.338 | | |
| 23 | | 44 | -.0110 | 65 | -22.55 | | |

## Rajan's Solution

| $\phi$ | $M_s$ | $\phi$ | $M_s$ | $\phi$ | $M_s$ | $\phi$ | $M_s$ |
|---|---|---|---|---|---|---|---|
| 5° | | 26 | | 47 | -.1049 | 67 | - 64.3 |
| 8 | | 29 | -.00035 | 50 | -.2311 | 69 | -110.5 |
| 11 | Less than | 32 | -.00039 | 53 | +.0253 | 71 | -104.4 |
| 14 | | 35 | +.0012 | 56 | +1.65 | 73 | + 79.8 |
| 17 | $10^{-4}$ | 38 | +.0063 | 59 | +4.70 | 75 | +616.0 |
| 20 | | 41 | +.0103 | 62 | +2.82 | | |
| 23 | | 44 | -.0118 | 65 | -23.5 | | |

Ex. 4
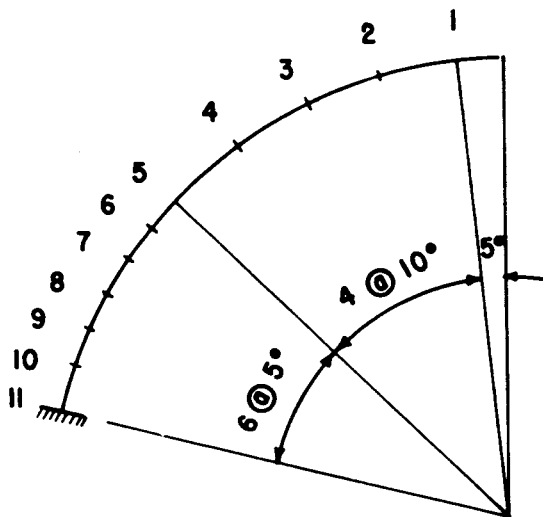


FIG. 13 - 10 CONE
ARRANGEMENT

The 75° sphere as in Ex. 1 was divided into 10 cones and a 5° cap as shown in Fig. 13. The analysis was obtained by a superposing edge displacements found from the membrane solution to obtain correct boundary conditions. The meridianal moments are listed below.

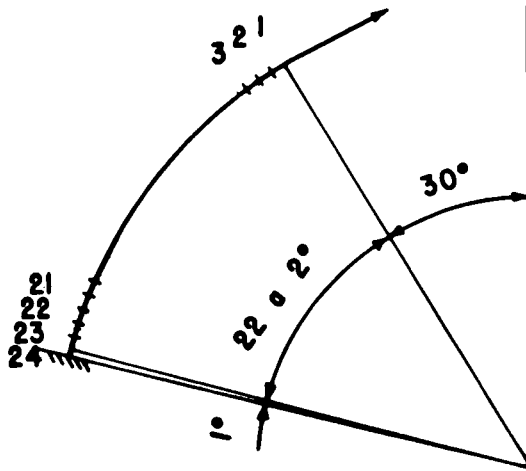| Joint | $M_s$ | Joint | $M_s$ |
|---|---|---|---|
| 1 | -12 | 7 | + 1 |
| 2 | - 2 | 8 | + 6 |
| 3 | -.7 | 9 | -23 |
| 4 | -.4 | 10 | -125 |
| 5 | -.2 | 11 | +594 |
| 6 | -.3 | | |

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Ex. 5

**N = 5000**



**FIG. 14-23 CONE ARRANGEMENTS WITH CAP REPLACED BY A MEMBRANE FORCE.**

The same $75^\circ$ sphere as in Ex. 1. A $30^\circ$ cap was removed and the remaining portion divided into 23 conical rings (Fig. 14). The force at the edge of the cap, known to be very close to membrane tension of $\frac{pa}{2}$, is applied in the direction of the tangent of the cap. The pressure over the remaining portion of the sphere was approximated at the joints as usual. The solution yields the following meridianal moments.

| Joint | $\phi$ | $M_s$ | Joint | $\phi$ | $M_s$ | Joint | $\phi$ | $M_s$ |
|-------|--------|-------|-------|--------|-------|-------|--------|-------|
| 1 | $30^\circ$ | 0 | 9 | 46 | -51.4 | 17 | 62 | -47.8 |
| 2 | 32 | -10.8 | 10 | 48 | -51.2 | 18 | 64 | -61.2 |
| 3 | 34 | -31.6 | 11 | 50 | -51.2 | 19 | 66 | -93.1 |
| 4 | 36 | -45.8 | 12 | 52 | -51.1 | 20 | 68 | -141.8 |
| 5 | 38 | -52.0 | 13 | 54 | -50.5 | 21 | 70 | -175.0 |
| 6 | 40 | -53.3 | 14 | 56 | -49.2 | 22 | 72 | -101.9 |
| 7 | 42 | -52.7 | 15 | 58 | -47.1 | 23 | 74 | +239.0 |
| 8 | 44 | -51.9 | 16 | 60 | -45.4 | 24 | 75 | +565.6 |

Ex. 6

The 75° sphere of Ex. 1 was divided in three different arrangements of cones, and analyzed with the approximate joint loads. The three cone arrangements are:

(1) 5° cap and 25 cones with subtended angles ranging from top to bottom of 20 of 3° and 5 of 2°. This is the arrangement used in Examples 1, 2, and 3.

(2) 5° cap and 35 cones with subtended angles of 5 of 1°, 5 of 2°, 10 of 3°, 10 of 2°, and 5 of 1°.

(3) 5° cap and 45 cones with subtended angles of 5 of 1°, 25 of 2°, and 15 of 1°.

The resulting meridianal moments for each case are plotted in Fig. 15.
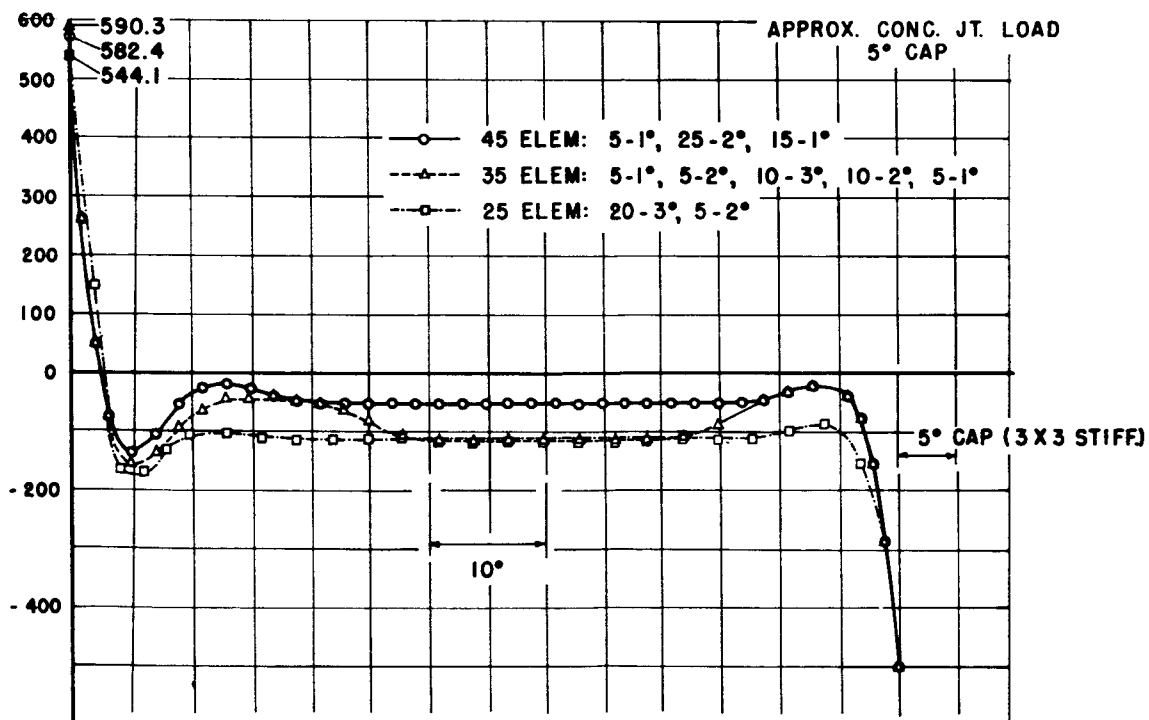


FIG. 15   M$_S$ DISTRIBUTION FROM SOLUTIONS
USING APPROXIMATE JOINT LOADS.

Ex. 7    By the same three cone arrangements as in Ex. 6, the shell was

analyzed with F.E.F. joint forces.  The resulting meridianal mom-
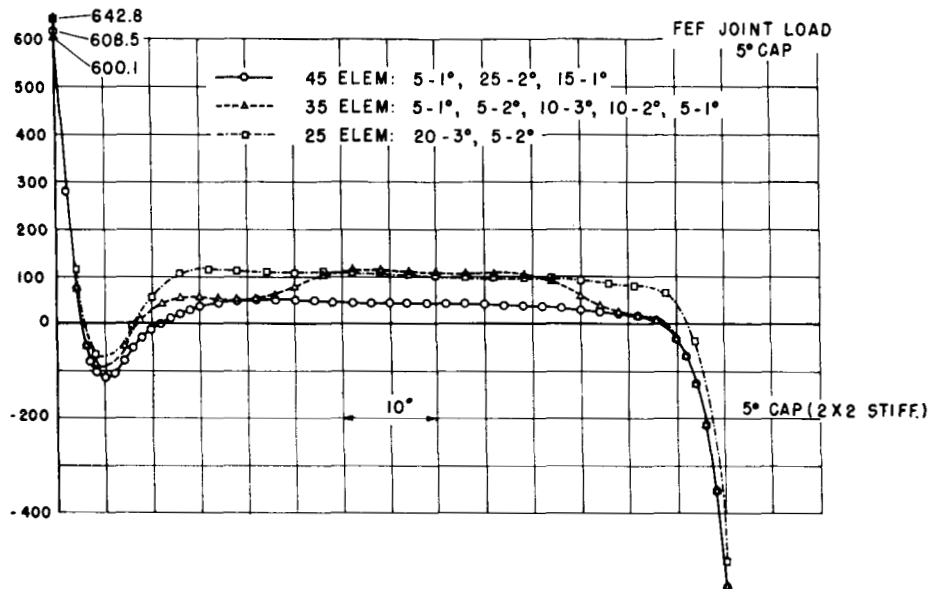
ents are plotted in Fig. 16.



**FIG. 16  M$_s$ DISTRIBUTION FROM SOLUTIONS
USING FEF JOINT LOADS.**

Ex. 8    Same three cone arrangements as in Exs. 6 and 7.  Superposition of

edge-displacement effects on membrane solutions.  The resulting $M_s$
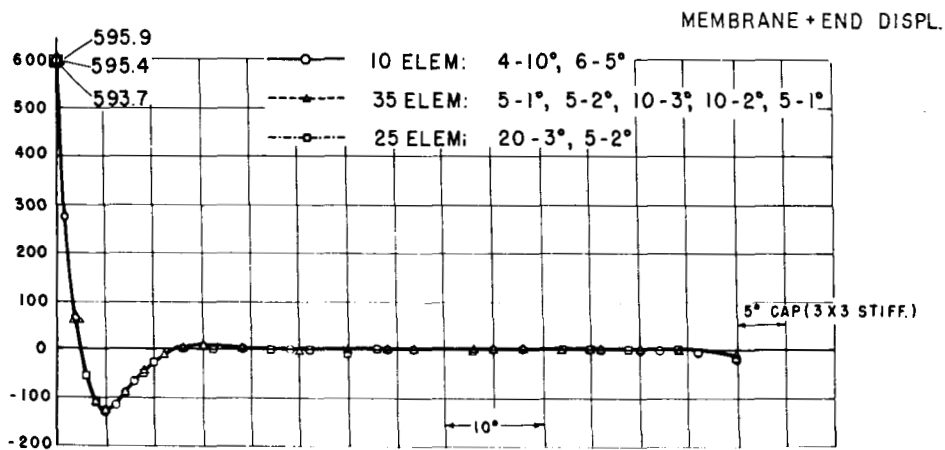
are plotted in Fig. 17.



**FIG. 17   M$_s$ DISTRIBUTION - END DISPLACEMENT EFFECT**

Ex. 9   The 75° sphere of Ex. 1 was analyzed with approximate joint loads

using three different cone arrangements which involve uniform

element lengths and break angles everywhere in the shell, except

in regions next to the cap.   The three cases are:

(1)   5° cap and 45 cones:   5 of 2°, 40 of $1\frac{1}{2}$°

(2)   5° cap and 35 cones at 2° spacing

(3)   5° cap and 24 cones:   1 of 1°, 23 of 3°
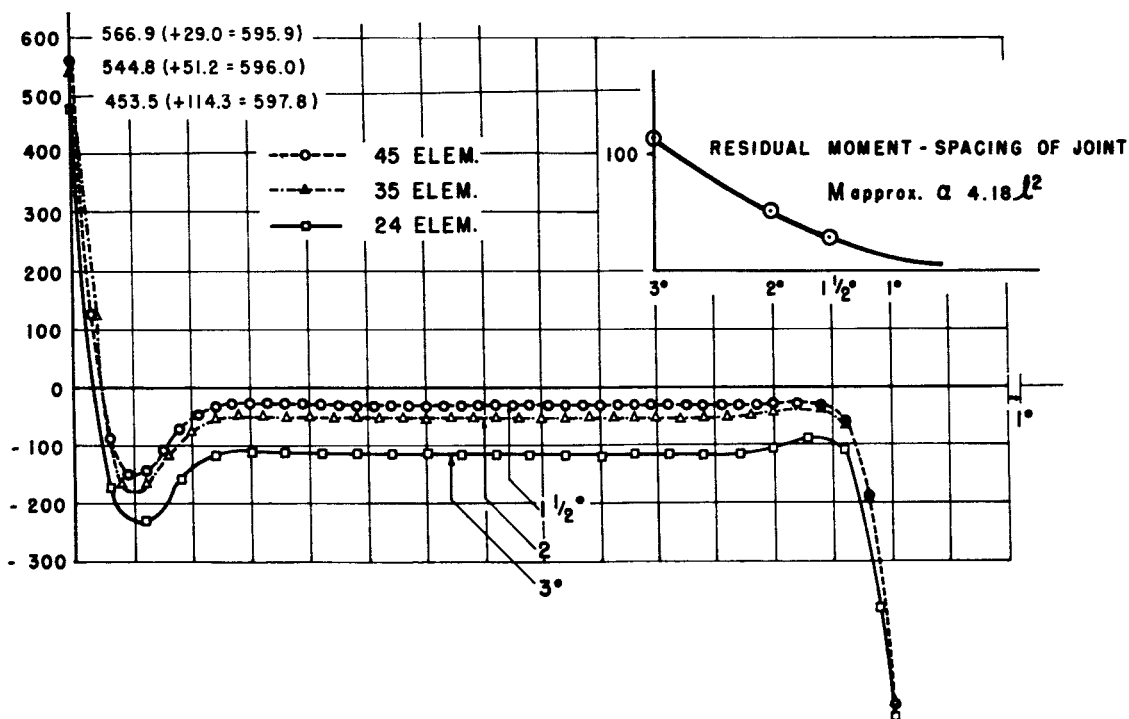
The meridianal moments are plotted in Fig. 18.



FIG. 18   M$_S$   DISTRIBUTION FROM SOLUTIONS USING
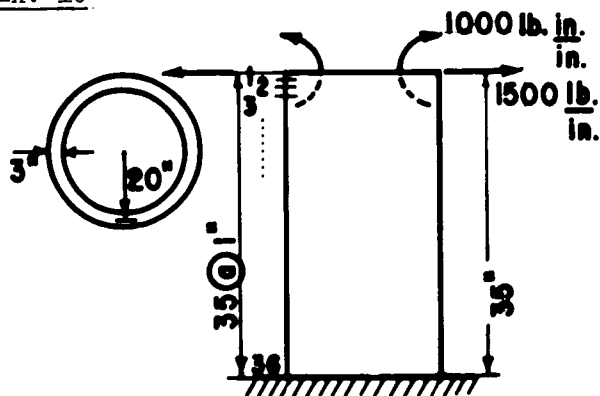EVEN SPACING OF JOINTS.

**FIG. 19**

A circular pipe 20" in diameter, 35" long, and 3" thick is subjected to a moment of 1000 lb-in/in and a horizontal force of 1500 lbs per in. at one end; the other end is fixed. Let $E = 3 \times 10^6$ psi, $\nu = 0$.

This problem was solved by a closed form mathematical procedure using 35 cylindrical elements of 1" length. The results of both solutions are tabulated below.

Pipe Solution by Mathematical Approach

| Joint | $M_x$ | Q | w | Joint | $M_x$ | Q | w | Joint | $M_x$ | Q | w |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1000 | 1500 | .0252 | 13 | 1083 | -302 | -.0018 | 25 | -150 | 10 | -.0002 |
| 2 | 2234 | 985 | .0206 | 14 | 801 | -261 | -.0019 | 26 | -138 | 14 | -.0001 |
| 3 | 3003 | 570 | .0163 | 15 | 561 | -219 | -.0018 | 27 | -122 | 17 | -.0001 |
| 4 | 3405 | 247 | .0125 | 16 | 363 | -178 | -.0017 | 28 | -104 | 18 | |
| 5 | 3525 | 5 | .0091 | 17 | 204 | -140 | -.0016 | 29 | - 86 | 19 | |
| 6 | 3439 | -167 | .0063 | 18 | 81 | -107 | -.0014 | 30 | - 67 | 18 | |
| 7 | 3210 | -282 | .0040 | 19 | - 11 | - 78 | -.0012 | 31 | - 49 | 18 | Less |
| 8 | 2890 | -350 | .0021 | 20 | - 76 | - 53 | -.0010 | 32 | - 31 | 18 | than |
| 9 | 2522 | -381 | .0007 | 21 | -120 | - 33 | -.0008 | 33 | - 13 | 17 | $10^{-4}$ |
| 10 | 2137 | -385 | -.0003 | 22 | -145 | - 18 | -.0006 | 34 | 4 | 17 | |
| 11 | 1759 | -369 | -.0010 | 23 | -156 | - 5 | -.0005 | 35 | 21 | 17 | |
| 12 | 1404 | -339 | -.0015 | 24 | -157 | 4 | -.0003 | 36 | 38 | 17 | |

## Pipe Solution by Finite Element Approach

| Joint | $M_x$ | Q | w | Joint | $M_x$ | Q | w | Joint | $M_x$ | Q | w |
|-------|-------|------|--------|-------|-------|------|--------|-------|-------|----|--------|
| 1 | 1000 | 1500 | .0252 | 13 | 1082 | -302 | -.0018 | 25 | -150 | 10 | -.0002 |
| 2 | 2234 | 985 | .0206 | 14 | 801 | -261 | -.0019 | 26 | -137 | 14 | -.0001 |
| 3 | 3003 | 570 | .0163 | 15 | 561 | -218 | -.0018 | 27 | -122 | 17 | -.0001 |
| 4 | 3405 | 247 | .0125 | 16 | 363 | -178 | -.0017 | 28 | -104 | 18 | |
| 5 | 3525 | 5 | .0091 | 17 | 204 | -140 | -.0016 | 29 | - 86 | 19 | |
| 6 | 3438 | -167 | .0063 | 18 | 81 | -107 | -.0014 | 30 | - 67 | 18 | |
| 7 | 3209 | -282 | .0040 | 19 | - 11 | - 78 | -.0012 | 31 | - 49 | 18 | |
| 8 | 2890 | -350 | .0021 | 20 | - 77 | - 53 | -.0010 | 32 | - 31 | 18 | |
| 9 | 2522 | -381 | .0007 | 21 | -120 | - 33 | -.0008 | 33 | - 13 | 17 | |
| 10 | 2137 | -385 | -.0003 | 22 | -145 | - 17 | -.0006 | 34 | 4 | 17 | |
| 11 | 1758 | -369 | -.0010 | 23 | -156 | - 5 | -.0005 | 35 | 21 | 17 | |
| 12 | 1403 | -339 | -.0015 | 24 | -157 | 4 | -.0003 | 36 | 38 | 17 | |

B. Computer Programs

For an elastic analysis of shells of revolution, the following individual programs (coded originally in Fortran language) are now available.

1. Geometry and Thompson functions for conical elements.

The argument for Thompson functions $(y_i, y_j)$ may be very large, and the Thompson functions will involve numbers much greater than $10^{38}$ or smaller than $10^{-38}$ which are beyond the capacity of IBM 7090. Fortunately, the flexibilities of a conical element always involve the product of a ber-series function and a ker-series function. This program yields Thompson functions in which a certain positive power is taken off from all the ber-series functions while an equal negative power is taken off from the ker-series functions. These Thompson functions of false power will yield the correct flexibility matrix of a cone just as well as Thompson functions of true power.

2. Flexibility and stiffness of conical elements, good for $\alpha \lesssim 90°$.

3. Thompson functions for individual numbers, good for arguments up to 120, output in true power.

4. Flexibility and stiffness of spherical cap.

    a. with singularity

    b. without singularity together with F.E.F. output

5. Flexibility and stiffness of cylindrical element (in subroutine form)

6. Approximate joint loads

7. Solution program using approximate joint loads and 3 x 3 cap stiffness, good for 45-element cone arrangement and for zero or known end-joint displacements. For other boundary conditions, it is necessary to regroup the unknowns and change the arrangements of several matrices involved.

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8. Solution program using 2 x 2 cap stiffness and F.E.F. joint loads. The F.E.F. joint loads are developed within the program. The output gives the final solution (sum of fixed cones solution and the effect of unbalanced joint loads). This is good for cases with the same boundary conditions as mentioned in 7.

9. A program to compute the remaining forces $(N_\theta, M_\theta)$ and stresses in the cones.

# V.  CONCLUSIONS

Examples 1 and 2 of the previous chapter showed that the solutions using approximate joint loads and the F.E.F. joint loads represent two opposite trends of slow convergence.  The former (using approximate loads) yields negative moments in the interior regions of the shell, and $N_j$ values in the conical elements smaller than the known membrane force, while the $N_i$ values are always greater than the $N_j$ values.  The latter approach (using F.E.F. joint loads) yields positive moment at interior joints and $N_j$ values greater than the known membrane force, while the $N_i$ values are always smaller than the $N_j$ values.  It is believed that the appearance of negative residual moments in the first approach is a natural consequence of the application of concentrated forces, and that the appearance of positive residual moment in the second approach is due to the fact that by the procedure used we have introduced some large quantities (the F.E.M.) into a region where these quantities do not exist.  Since the unbalanced moment around a joint is much smaller than the fixed-end moments and the distributed moments can never exceed the unbalanced moments in magnitude, positive moments remain at the joints.  However, as the number of elements becomes very large and the element lengths become small, these residual moments, positive or negative, do diminish.  Examples 6 and 7 showed that the results of analysis by both approaches converge.  Examples 1, 2, 3 point out the fact that the joint displacements obtained by different procedures are about the same.

Example 5 illustrated the use of a short-cut method by which a 23-element solution yields nearly as good results as a regular solution using 45 elements (Ex. 6).

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Examples 6 and 7 showed that whenever joint loads are applied some residual moments always remain in regions where the stress condition is known to be membrane. These residual moments are positive when F.E.F. joint loads are used and negative when approximate joint loads are used. Their magnitudes depend on the lengths of the elements and the break angle between the elements. When the element lengths and break angles are made smaller, the residual moments diminish quickly $(M \propto \ell^2)$. It appears that by averaging the results of a solution using approximate joint loads and those of another solution using F.E.F. joint loads, both based on the same arrangement of elements, the correct results can be obtained. Such a step, however, needs rational justification.

Examples 4 and 8 indicate the advantage of superposition of edge displacement effects on a membrane solution. By this procedure a 10-element arrangement yields nearly as good results as does a 25-element or a 35-element arrangement. No problems of residual moments arise.

Example 9 showed that when constant element lengths and break angles are used throughout (except near vertex), the $M_s$ distribution using various element lengths all take the same shape. Their plots appear only at different heights from a base line in a graph. These different heights from the base line represent the residual moments for the particular element lengths and break angles used in the analysis. Comparing Examples 8 and 9, we see that the correct $M_s$ distribution can be obtained by simply shifting the base line down (for approximate joint loads) and up (for F.E.F. joint loads) through an amount equal to the residual moment and read the curve referred to the new base line. But if the lengths of the elements are kept the same while the break angles are not (possible near boundaries), such an approach should not be taken because the non-uniformity

of break angles will cause local change of the slope of the $M_s$ curve and thus impair the correctness of the result.

Example 10 simply showed the correctness and accuracy of using cylindrical elements for a solution.

We thus came to the following conclusions.

1. Finite element analysis of axisymmetrical shells of revolution does give correct results, but due to the space nature of the problem, a much larger number of elements than that needed for a frame analysis is required for the desired accuracy.

2. The use of F.E.F. joint loads, although exact in concept, offers no practical advantage over the use of approximate joint loads. On the contrary it involves more steps for a solution than does the use of approximate joint loads.

3. For possible cases, a solution by superposition of edge-displacement effects and membrane effects is always recommended. Such a solution can make use of a relatively small number of elements and yet yield accurate results. For shells with arbitrary shape and thickness, a finite-element membrane solution is fairly simple to formulate.

4. The use of equal element lengths and equal break angles throughout the shell is recommended. In such cases a relatively small number of elements could yield correct results by the use of shifting base line approach discussed before.

5. Short-cut methods of replacing a portion of the shell with known forces can be used to save effort if possible.

6. Joint displacements are relatively insensitive to the solution method and the number of elements used.

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